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A survey on t-core partitions

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In memory of Srinivasa Ramanujan

Abstract. t-core partitions have played important roles in the theory of partitions and related areas. In this survey, we briefly summarize interesting and important results on t-cores from classical results like how to obtain a generating function to recent results like simultaneous cores. Since there have been numerous studies on t-cores, it is infeasible to survey all the interesting results. Thus, we mainly focus on the roles of t-cores in number theoretic aspects of partition theory. This includes the modularity of t-core partition generating functions, the existence of t-core partitions, asymptotic formulas and arithmetic properties of t-core partitions, and combinatorial and number theoretic aspects of simultaneous core partitions. We also explain some applications of t-core partitions, which include relations between core partitions and self-conjugate core partitions, a t-core crank explaining Ramanujan's partition congruences, and relations with class numbers.

Keywords. t-core, modular equation, modular forms, simultaneous core, crank, lattice path

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1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n is a non-increasing sequence of natural numbers whose sum is n. Let $|\lambda|$ denote the size of λ, $\sum \lambda_i$. Each λ_i is called a part of λ. Partitions are represented as Ferrers graphs (or Young diagrams), where the summands in the partition are arranged in rows. For example, the below Ferrers graph is for $\lambda = (8, 6, 2, 1)$ of 17 (see Figure 1). The conjugate of a partition λ, denoted by λ^c, is the partition whose Ferrers graph is the reflection of the Ferrers graph of λ along the diagonal. A partition λ is called self-conjugate if $\lambda = \lambda^c$.

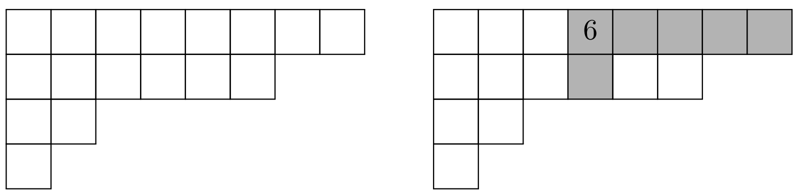

Figure 1: The Ferrers graph of the partition $(8, 6, 2, 1)$ and a hook length

For a given partition λ, a box at (i, j) in the Ferrers diagram of λ is the box in the ith row from the top and the jth column from the left. The *hook* of a box at (i, j) contains the boxes at $\{(k, j) \mid k \geq i\} \cup \{(i, k) \mid k \geq j\}$. The *hook length* $h_{i,j}$ of a box at (i, j) is the number of boxes in the hook containing that box. In other words, the hook length is the number of boxes directly right and directly below the box, plus 1 for the box itself.

For a positive integer t, a partition is called a t-core if none of the hook lengths are multiples of t. The name *t-core* comes from that it is the remaining part after deleting t-rim hooks as we will see in Section 2.A. soon.

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It is doubtful that Ramanujan knew the notion of t-core partitions, but the study of t-core partitions is deeply related with the works of Ramanujan. Let $p(n)$ be the number of partitions of n. Ramanujan's striking three congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

are arguably some of the most impactful results in the theory of partitions [And88]. In particular, "Ramanujan's most beautiful identity" (in the words of G.H. Hardy [Ram00, p. xxxv]) states that

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

where here and in the sequel, we will use the following standard q-series notation:

$$\begin{aligned} (a;q)_0 &:= 1, \\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n\geq 1, \\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n,\ |q|<1. \end{aligned}$$

The "Ramanujan's most beautiful identity" can be proven from the fact that a generating function for 5-core partitions is a Hecke eigenform. More specifically, let $f_5(q)$ be a generating function for the number of 5-core partitions of n, $c_5(n)$,

$$f_5(q) := \sum_{n\geq 0} c_5(n)q^{n+1} = q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}$$

and U_p be an operator defined by the q-expansion $f|U_p = \sum a(pn)q^n$ for $f = \sum a(n)q^n$. In Section 2, we give more details on how to obtain a generating function. We also introduce modular forms and operators on them. The theory of modular forms implies that $f_5|U_5 = 5f_5$ which is equivalent to

$$\begin{aligned} q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}\Big|U_5 &= 5q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \\ \left((q^5;q^5)_\infty^5\sum_{n\geq 0}p(n)q^{n+1}\right)\Big|U_5 &= 5q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \\ (q;q)_\infty^5\sum_{n\geq 1}p(5n-1)q^n &= 5q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \\ \sum_{n\geq 0}p(5n+4)q^n &= 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \end{aligned}$$

In this sense, Ramanujan's partition congruence $p(5n+4) \equiv 0 \pmod{5}$ follows from the fact that $c_5(5n+4) \equiv 0 \pmod{5}$ and this connection plays an important role to construct the t-core crank in Garvan, Kim, and Stanton's work [GKS90].

Moreover, Ramanujan's modular equations imply many relations among t-core partitions. For example, Baruah and Berndt [BaBe07, Theorem 4.1] obtained a linear relation between the number of 3-core partitions of n, $c_3(n)$,

$$c_3(4n+1) = c_3(n)$$

from Ramanujan's modular equation [Ber91, Entry 5(i), p. 230]

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1,$$

where β has degree 3 over α. (See Section 3.B. for the definition of a modular equation, α, β, and the degree.)

Core partitions significantly contribute in various areas of mathematics. In this survey, we will focus on partition theoretic aspects of t-core partitions and connections with other number theoretic objects. For other aspects of t-core partitions, one may see representation theory of the symmetric group [GrOn96, JaKe81, Ols93] and symmetric functions [Sta99].

The paper is organized as follows. In Section 2, we explain how to obtain a generating function for the number of t-core partitions and the modularity of a generating function. In Section 3, we survey results on t-core partitions. These include the existence of t-core partitions, linear relations among t-cores from Ramanujan's modular equations, asymptotic formulas to estimate the growth of t-core partitions, and arithmetic properties of t-cores. In Section 4, we introduce simultaneous core partitions, which have recently investigated extensively. In Section 5, we give some applications of t-cores in the theory of partitions and number theory. Among many, we choose a relation between core partitions and self-conjugate core partitions, Ramanujan's partition congruences for t-core cranks, and class numbers of quadratic forms. We conclude the survey with some remarks in Section 6.

2. Generating function and modularity

In this section, we explain how to find a generating function for t-core partitions and the modularity of a generating function.

2.A. Generating function

For a partition λ, it is clear that if λ has a hook length t, then λ is not a t-core partition. Therefore, we construct a t-core partition from λ by removing boxes of hook length t. From [JaKe81, Section 2.7], it is known that we can remove a box of hook length t by deleting a rim hook of that box. A rim hook of a box is determined from boxes of the rim in the Ferrers graph of λ between the two ends of the hook of that box.

Example 2.1. *Let $\lambda = (8, 6, 2, 1)$. In Figure 2, shaded areas are the hook and the corresponding rim hook of a box at $(1, 4)$ in λ.*

11	9	7	6	5	4	2	1
8	6	4	3	2	1		
3	1						
1							

11	9	7	6	5	4	2	1
8	6	4	3	2	1		
3	1						
1							

Figure 2: The hook and the corresponding rim hook of the partition $(8, 6, 2, 1)$

To construct a t-core partition from λ, it may be necessary to remove multiple rim hooks. We call the resulting partition $\lambda^{(t)}$, the t-core of λ.

Example 2.2. *Let $\lambda = (8, 6, 2, 1)$. Since the partition λ has a box of hook length 6, we remove a rim hook of length 6 from λ. There are two ways of removing a rim hook of length 6 from λ as shown* *in Figure 3. The new partition obtained by deleting a rim hook of length* 6 *from* λ *still has a box of hook length* 6*, so we proceed the same procedure until we remove all rim hooks of length* 6*. As one can see, the order in which one removes rim hooks does not change the resulting core partition* $\lambda^{(6)} = (2,1,1,1)$*.*

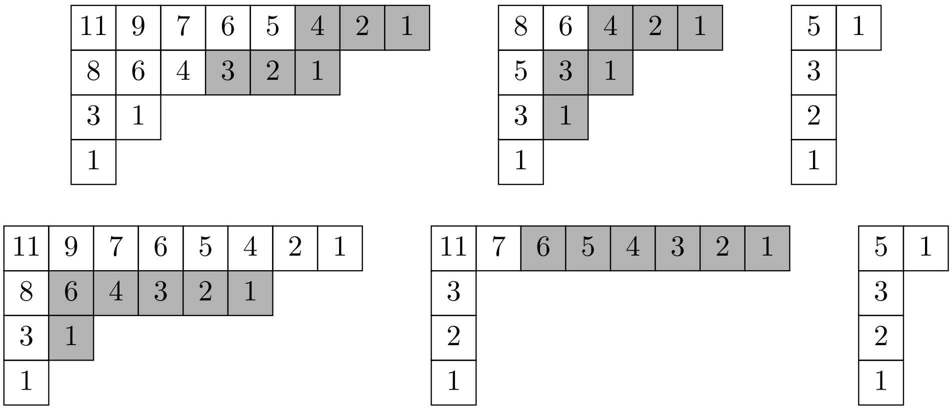

Figure 3: The process of obtaining the 6-core of the partition $(8,6,2,1)$

To simplify the removing process, we use the abacus diagram from James [Jam78]. The t-abacus diagram is a diagram with infinitely many rows labeled by $0,1,2,\dots$ and t columns labeled by $0,1,\dots,t-1$. We label the positions by $0,1,2,\dots$ from left to right and from bottom to top. For a partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_\ell)$, we denote the *beta-set* of λ, $\beta(\lambda) = \{\beta_1,\beta_2,\dots,\beta_\ell\}$, the set of the first column hook lengths of λ. We now define the t-abacus of λ, which is obtained from the t-abacus diagram by placing a bead on each position labeled by the elements in $\beta(\lambda)$. A position without bead is called a spacer.

Example 2.3. *Let* $\lambda = (11,10,9,6,4)$ *and* $\beta(\lambda) = \{15,13,11,7,4\}$*. The* 6*-abacus of* λ *is given in Figure 4.*

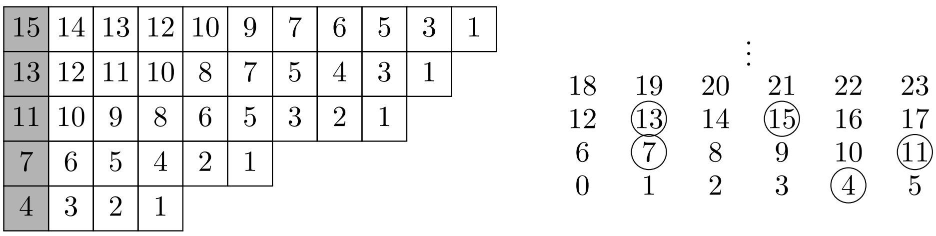

Figure 4: The 6-abacus of the partition $(11,10,9,6,4)$

The following lemma is a key to find $\lambda^{(t)}$.

Lemma 2.4. *[JaKe81, Lemma 2.7.13] For a partition* $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_\ell)$*, there exists a hook of length* t *in* λ *if and only if there exists* $\beta_i \in \beta(\lambda)$ *such that* $\beta_i > t$ *and* $\beta_i - t \notin \beta(\lambda)$*. Furthermore,* $\beta(\lambda) \cup \{\beta_i - t\} \setminus \{\beta_i\}$ *is the beta-set of* μ*, where* μ *is a partition which is constructed from* λ *by removing a rim hook of length* t*.*

By Lemma 2.4, the action to remove a rim hook of length t in the Ferrers graph of λ is equivalent to the action to slide down a bead in the t-abacus of λ. Moreover, if there is no $\beta_i \in \beta(\lambda)$ such that $\beta_i > t$ and $\beta_i - t \notin \beta(\lambda)$, λ is a t-core partition. Therefore, $\lambda^{(t)}$ is determined by sliding down all beads in the t-abacus of λ. In the above example, $\beta(\lambda^{(6)})$ is given by $\{7, 5, 4, 3, 1\}$ and then $\lambda^{(6)} = (3, 2, 2, 2, 1)$.

Let $(\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(t-1)})$ be the *t-quotient* of λ, where $\lambda_{(i)}$ is the partition whose jth part is k if there are k spacers below the jth bead from the bottom in column i. We note that $\sum_{i=0}^{t-1} |\lambda_{(i)}|$ is the number of removed rim hooks in the Ferrers graph of λ. In the above example, $\lambda_{(1)} = (1, 1)$, $\lambda_{(3)} = (2)$, $\lambda_{(5)} = (1)$, and $\lambda_{(0)} = \lambda_{(2)} = \lambda_{(4)} = \emptyset$.

From [JaKe81, Theorem 2.7.30], a partition λ is uniquely determined by its t-core $\lambda^{(t)}$ and its t-quotient $(\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(t-1)})$. Therefore, there exists a bijection $\phi : \mathcal{P} \to \mathcal{P}_t \times \mathcal{P}^t$ by $\phi(\lambda) = \{\lambda^{(t)}, (\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(t-1)})\}$ with $|\lambda| = |\lambda^{(t)}| + t\sum_{i=0}^{t-1} |\lambda_{(i)}|$. The injectivity of the map ϕ comes immediately from the definition. However, it takes time to check the surjectivity of this map thoroughly. We leave it to readers (see [JaKe81, Section 2.7]).

This bijection can be expressed as the following generating function identity. Let $c_t(n)$ be the number of t-core partitions of n, then a generating function of $c_t(n)$ is

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}. \tag{2.1}$$

In [GKS90, Section 2], Garvan, Kim, and Stanton gave an alternative proof of (2.1). They also found a generating function of self-conjugate t-core partitions (see [GKS90, Section 7]).

Let $sc_t(n)$ be the number of self-conjugate t-core partitions of n. Then,

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = (-q; q^2)_\infty (q^{4t}; q^{4t})_\infty^t \quad \text{and} \quad \sum_{n=0}^{\infty} sc_{2t+1}(n)q^n = \frac{(-q; q^2)_\infty (q^{4t+2}; q^{4t+2})_\infty^t}{(-q^{2t+1}; q^{4t+2})_\infty}. \tag{2.2}$$

2.B. Modularity

Most arithmetic properties of t-core partitions follow from the modularity of a generating function. To see this, we first let $\delta_t = \frac{t^2-1}{24}$ and $\eta(z)$ be the Dedekind's eta function defined by $\eta(z) = q^{\frac{1}{24}}(q; q)_\infty$, where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$, the upper half-plane. Then, we can rewrite a generating function for t-cores as an eta-quotient

$$\sum_{n \geq 0} c_t(n)q^{n+\delta_t} = \frac{\eta^t(tz)}{\eta(z)}.$$

The Dedekind's eta function is an example of weight 1/2 modular form and the modularity of eta-quotients has been studied extensively. Once we establish the modularity of a generating function, one employs various systematic techniques from the theory of modular forms. For the sake of brevity, we only briefly introduce modular forms of integral weight. For additional basic properties of modular forms, see [Ono04, Chaps. 1, 2, and 3].

Define $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$. Let $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) denote the vector space of holomorphic forms (resp. cusp forms) of weight k. Let $\mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$) denote the vector space of holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with a character χ.

For a prime p, we need to define the U_p-operator and the Hecke operator T_p on $\mathcal{M}_k(\Gamma_0(N), \chi)$. If $f(q)$ has the Fourier expansion $f(q) = \sum a(n)q^n$, then

$$U_p f := \sum a(pn)q^n \quad \text{and} \quad T_p f := \sum \left(a(pn) + a(n/p)\chi(p)p^{k-1} \right) q^n.$$

It is a standard fact that $T_p f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$) if $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \chi)$). We say that $f(z)$ is an eigenform of T_p if there is a $\lambda_p \in \mathbb{C}$ such that $T_p f = \lambda_p f$. We call $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$ a Hecke eigenform if $f(z)$ is an eigenform of T_p for all primes $p \nmid N$. The space of cusp forms $\mathcal{S}_k(\Gamma_0(N), \chi)$ has a subspace $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ and we call the Hecke eigenforms in $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ newforms. Throughout this paper, we assume that each newform $g(z)$ is normalized so that the Fourier coefficient of q is 1 in $g(z)$. It is well-known that $\mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ has a basis consisting of newforms, and if $g(z) \in \mathcal{S}_k^{new}(\Gamma_0(N), \chi)$ is a newform and $g(z)$ has a Fourier expansion of the form $g(z) = \sum_{n=1}^{\infty} b(n)q^n$, then $|b(p)| \leq 2p^{\frac{k-1}{2}}$ for all primes p. A special case of this bound was first proposed by Ramanujan [Ram1916], which claims that $|\tau(p)| \leq 2p^{11/2}$, where $\tau(n)$ is the Ramanujan's tau function. A general bound was first proved by P. Deligne [Del75]. For t-core partition generating functions, one can check that (see [Ono04, Section 1.4] for example),

$$f_p = \frac{\eta^p(pz)}{\eta(z)} \in \mathcal{M}_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$$

for a prime $p > 3$ and $\chi_p = \left(\frac{\cdot}{p}\right)$ is a Legendre symbol. For a non-prime t, one needs some extra works to see the modularity. For example, a 4-core partition generating function

$$\sum_{n\geq 0} c_4(n)q^{8n+5} = \frac{\eta^4(32z)}{\eta(8z)}$$

is a modular form of weight $3/2$ and $c_4(n)$ can be written in terms of a class number [OnSz97]. As we will see later, the modularity of generating functions is a crucial object to learn asymptotic behaviors of core partitions, linear relations, and more.

3. Results on t-cores

In this section, we start with the discussion on the existence of t-core partitions. In Section 3.2, we give relations among t-core partitions derived from Ramanujan's modular equations. In Sections 3.3 and 3.4, we discuss how fast the number $c_t(n)$ grows and investigate some arithmetic properties of them.

3.A. Existence of t-cores

Due to a connection to representation theory (see [GrOn96, JaKe81]), it is natural to ask whether there is a t-core partitions of n. The positivity of $c_t(n)$ is not clear at all from its generating function. For $t = 2$, from the Gauss identity, we find that

$$\sum_{n\geq 0} c_2(n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \sum_{n\geq 0} q^{n(n+1)/2},$$

which says that $c_2(n) > 0$ if and only if n is a triangular number. For $t = 3$, a generating function is essentially an Eisenstein series [GrOn96, Section 3, p. 340],

$$\sum_{n\geq 0} c_3(n)q^{3n+1} = \frac{\eta^3(9z)}{\eta(3z)} = \sum_{n\geq 0} \sigma_3(n)q^n,$$

where $\sigma_3(n)$ is defined by

$$\sigma_3(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3 \\ \sum_{d|n} \left(\frac{d}{3}\right), & \text{otherwise.} \end{cases}$$

This implies that $c_3(n) = \sigma_3(3n+1)$, which gives that $c_3(n) > 0$ if and only if there is no prime $p \equiv 2 \pmod 3$ with odd $\mathrm{ord}_p(3n+1)$, where $\mathrm{ord}_p(3n+1)$ is the exponent of the largest power of p that divides $3n+1$. While $c_2(n)$ and $c_3(n)$ might be zero, it had been conjectured that $c_t(n) > 0$ for $t \geq 4$. The first few cases were proven by Erdmann and Michler [ErMi77] and by Ono [Ono94, Ono95] case by case. Finally, Granville and Ono [GrOn96, Theorem 1] succeeded to prove the positivity conjecture for the number of t-core partitions.

Theorem 3.1. *If $t \geq 4$, then $c_t(n) > 0$ for every nonnegative integer n.*

The key ingredient of the proof is a new generating function of Garvan, Kim, and Stanton [GKS90], which gives

$$c_t(n) = \left| \left\{ (x_0, x_1, \ldots, x_{t-1}) \in \mathbb{Z}^t : n = \frac{t}{2}\sum_{i=0}^{t-1} x_i^2 + \sum_{i=0}^{t-1} i x_i \quad \text{with } x_0 + x_1 + \cdots + x_{t-1} = 0 \right\} \right|.$$

For $t \geq 17$, they proved the positivity by finding a solution for the above quadratic equation. For $t < 17$, they proved the positivity by employing the theory of modular forms.

In the case of self-conjugate t-cores, Baldwin, Depweg, Ford, Kunin, and Sze [BDFKS06, Theorem 1] proved the existence.

Theorem 3.2. *For $t = 8$ or $t \geq 10$, $sc_t(n) > 0$ for all integers $n > 2$.*

When $t \leq 7$ or $t = 9$, there is a positive integer n such that $sc_t(n) = 0$. For example, one can check that $sc_9(n) = 0$ for a positive integer k and $n = \frac{4^k - 10}{3}$ [BDFKS06, Proposition 14].

3.B. Modular equation and t-cores

Ramanujan recorded lots of modular equations in his notebooks and lost notebook (see [Ber91, Ram88]). Various relations on t-cores follow from Ramanujan's modular equations. Before stating t-core partition results, we first give a definition of modular equations briefly. We recommend Berndt's book [Ber06, Chapter 6] for an introduction to the theory of modular equations. The complete elliptic integral of the first kind associated with the modulus k, $0 < k < 1$, is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The complementary modulus k' is defined by $k' := \sqrt{1-k^2}$ and we set $K' := K(k')$. Let K, K', L, and L' be the complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. If

$$n\frac{K'}{K} = \frac{L'}{L} \tag{3.3}$$

holds for a positive integer n, then a modular equation of degree n is a relation between the moduli k and ℓ that is implied by (3.3). Ramanujan recorded modular equations using $\alpha = k^2$ and $\beta = \ell^2$. We say β has degree n over α. Modular equations are deeply related with q-series and many important q-series can be expressed in terms of modular equations. For example, if $q = \exp(-\pi K'/K)$, $f(-q) = (q;q)_\infty$, $z_\alpha := {}_2F_1(1/2, 1/2; 1; \alpha)$, β has degree n over α, and $z_\beta := {}_2F_1(1/2, 1/2; 1; \beta)$, then

$$f(-q) = \frac{\sqrt{z_\alpha}\alpha^{1/24}(1-\alpha)^{1/6}}{2^{1/6}q^{1/24}} = \frac{\sqrt{z_\beta}\beta^{1/24}(1-\beta)^{1/6}}{2^{1/6}q^{1/24}},$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric series defined by

$${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)} z^n.$$

Baruah and Berndt [BaBe07] obtained several core partition identities from Ramanujan's modular equations. Beside the one in the introduction, we give one more example. Baruah and Berndt [BaBe07, Theorem 4.2] proved that

$$c_5(4n+3) = c_5(2n+1) + 2c_5(n) \tag{3.4}$$

from modular equations of degree 5 [Ber91, p. 280]. We remark that Golze [Gol16] combinatorially proved that

$$c_3(n) = c_3\left(p^k n + \frac{p^k-1}{3}\right)$$

for every prime $p \equiv 2 \pmod 3$ and positive even integers k.

On the other hand, Berkovich and Yesilyurt [BeYe08, Theorem 1.1] proved that

$$c_7(2n+2) > 2c_7(n) \tag{3.5}$$

from modular equations of degree 7 [Ber91, Entry 19, p. 314]:

$$\left(\frac{1}{2}(1+(\alpha\beta)^{1/2}+[(1-\alpha)(1-\beta)]^{1/2})\right)^{1/2} = 1-(\alpha\beta(1-\alpha)(1-\beta))^{1/2},$$
$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = \frac{z_\alpha}{z_\beta}\left(\frac{1}{2}(1+(\alpha\beta)^{1/2}+[(1-\alpha)(1-\beta)]^{1/2})\right)^{1/2}.$$

Linear relations and the above inequality follow also from the theory of modular forms. In particular, when t is small, the dimensions of the spaces of modular forms containing the t-core partition generating function are small and thus one can easily find the image of Hecke operators which is very helpful to obtain these type of relations. For example, $f_5 = \eta^5(5z)/\eta(z) \in \mathcal{M}_2(\Gamma_0(5), \chi_5)$ and $\dim\mathcal{M}_2(\Gamma_0(5), \chi_5) = 1$, therefore one can readily find that $T_2(f_5) = f_5$ by matching the first coefficient of q-expansions. Therefore, $T_2(T_2 f_5) = f_5$ which implies (3.4). Similarly, $f_7 = \eta^7(7z)/\eta(z) \in \mathcal{M}_3(\Gamma_0(7), \chi_7)$ and $\dim\mathcal{M}_3(\Gamma_0(7), \chi_7) = 3$. Moreover one can find that

$$\left\{\frac{\eta^7(z)}{\eta(7z)}, \eta^3(z)\eta^3(7z), \frac{\eta^7(7z)}{\eta(z)}\right\}$$

is a basis for $\mathcal{M}_3(\Gamma_0(7), \chi_7)$. Since $\frac{\eta^7(z)}{\eta(7z)}$ is an Eisenstein series and $\eta^3(z)\eta^3(7z)$ is a CM form, one can find the exact formulas from the literature, which leads the inequality (3.5) and more. For more details, one may see [GKS90, GrOn96, Kim10, KiRo14]. One can also systematically obtain modular equations involving core partition generating functions using the theory of modular functions. For this direction, one may consult with Park [Par14] for instance.

3.C. Asymptotic results

Since Hardy and Ramanujan obtained an asymptotic formula for the partition function, there have been numerous studies on how fast partition functions grow. The growth of t-core partitions is particularly interesting in the sense that the existence of t-core partitions and inequalities among core partitions have been notoriously hard to prove. Using the circle method, Anderson [And08, Theorem 2] obtained an asymptotic formula: for $t \geq 6$,

$$c_t(n) = \frac{(2\pi)^{(t-1)/2} A_t(n)}{t^{t/2}\Gamma((t-1)/2)}(n+(t^2-1)/24)^{(t-3)/2} + O(n^{(t-1)/4}),$$

where $\Gamma(x)$ is the gamma function and $A_t(n)$ is an explicit sum involving 24th roots of unity coming from the transformation formula for the Dedekind eta function. For precise definition, see [And08]. One of her goals was to prove Stanton's conjecture of which general case remain open until now.

Conjecture 3.3. *[Sta99] If $t \geq 4$ and $n \neq t+1$, then $c_{t+1}(n) \geq c_t(n)$.*

Anderson's asymptotic formula implies that Stanton's conjecture is true asymptotically.

Theorem 3.4. *[And08, Theorem 1] If $4 \leq t_1 \leq t_2$, then $c_{t_1}(n) < c_{t_2}(n)$ for sufficiently large n.*

Granville and Ono [GrOn96] used the fact that

$$f_p = \frac{\eta^p(pz)}{\eta(z)} = e_p E_p(z) + \sum a_i g_i(z),$$

where e_p is the constant defined by

$$\frac{1}{e_p} = \frac{\left(\frac{p-3}{2}\right)! p^{\frac{p}{2}}}{(2\pi)^{\frac{p-1}{2}}} L\left(\frac{p-1}{2}, \chi_p\right),$$

$E_p(z)$ is an Eisenstein series, and $g_i(z)$ are newforms in the space $\mathcal{S}^{new}_{(p-1)/2}(\Gamma_0(p), \chi_p)$. As Fourier coefficients of Eisenstein series $E_p(z)$ is known and the coefficients of newforms are bounded by the result of Deligne, the above decomposition gives the following formula

$$c_p(n-(p^2-1)/24) = a_p \sum_{d|n} \left(\frac{n}{p}\right)\left(\frac{n}{d}\right) d^{(p-3)/2} + O(n^{(p-1)/4+\epsilon}).$$

Later, Kim and Rouse [KiRo14, Theorem 1.1] made the above estimate effective. They also obtained an effective version of Anderson's asymptotic formula and confirmed that Stanton's conjecture is true up to $t = 198$ [KiRo14, Corollary 1.6].

For self-conjugate partitions, Hanusa and Nath made an analogous conjecture corresponding to Stanton's conjecture.

Conjecture 3.5. *[HaNa13, Conjectures 1.1 and 1.2] Let $t \geq 9$ or $t = 6, 8$, then $sc_{t+2}(n) \geq sc_t(n)$ for all $n \geq 20$ (resp. 56) if t is even (resp. odd).*

Alpoge [Alp14, Theorem 3] obtained an asymptotic formula for the number of self-conjugate t-core partitions by employing the circle method.

$$sc_t(n) = \frac{(2\pi)^{t/4} B(t)}{(2t)^{t/4}\Gamma(t/4)}(n+(t^2-1)/24)^{t/4-1} + O(n^{t/8}), \quad \text{if } t \text{ is even,}$$
$$sc_t(n) = \frac{(2\pi)^{(t-1)/4} C(t)}{(2t)^{(t-1)/4}\Gamma((t-1)/4)}(n+(t^2-1)/24)^{(t-1)/4-1} + O(n^{(t-1)/8}), \quad \text{if } t \text{ is odd.}$$

Using the above asymptotic formulas, Alpoge [Alp14, Theorem 2] confirmed that Hanusa and Nath's conjecture holds for sufficiently large n.

3.D. Arithmetic of t-cores

Inspired by Ramanujan's three congruences, congruence properties of $c_t(n)$ have been established. Garvan, Kim, and Stanton [GKS90, Corollary 1] showed that, for a positive integer a, a nonnegative integer n, and for each $\ell = 5, 7, 11$,

$$c_\ell(\ell^a n - \delta_\ell) \equiv 0 \pmod{\ell^a}$$

with $\delta_\ell = \frac{\ell^2-1}{24}$. Granville and Ono [GrOn96, Proposition 3] found similar congruence relations using Ramanujan's congruences. With the same setting that Garvan, Kim, and Stanton had, they proved that

$$\begin{aligned} c_{5^a}(5^a n - \delta_{5,a}) &\equiv 0 \pmod{5^a}, \\ c_{7^a}(7^a n - \delta_{7,a}) &\equiv 0 \pmod{7^{\lfloor \frac{a}{2} \rfloor + 1}}, \\ c_{11^a}(11^a n - \delta_{11,a}) &\equiv 0 \pmod{11^a}, \end{aligned}$$

where $\delta_{\ell,a} \equiv \frac{1}{24} \pmod{\ell^a}$. Using the technique that Ono [Ono00] developed, Chen [Che09, Theorem 2] proved that, for all two coprime integers m and n,

$$c_{2^t}\left(\frac{mn - \frac{4^t-1}{3}}{8}\right) \equiv 0 \pmod{2^\ell},$$

where t and ℓ are positive integers and m is a square-free odd integer that has at least $\frac{\ell(4^t-1)}{3}$ prime divisors.

Hirschhorn and Sellers [HiSe99] conjectured that, for positive integers $t \geq 2$, $k = 0, 2$, and all positive integers n,

$$c_{2^t}\left(\frac{3^{2^{t-1}-1}(24n + 8k + 7) - \frac{4^t-1}{3}}{8}\right) \equiv 0 \pmod{2}.$$

The conjecture was proved by Hirschhorn, Kolitsch, and Sellers [HiSe96, HiSe99, KoSe99] when $t = 2, 3, 4$ and fully proved by Chen [Che13, Theorem 2]. Applying Hecke operators on some space of modular forms, Boylan [Boy02, Theorem 1.1] proved that, for any positive integer t and distinct odd primes $p_1, p_2, \ldots, p_{\frac{4^t-1}{3}}$ such that $\gcd(n, \prod p_i) = 1$,

$$c_{2^t}\left(\frac{p_1 p_2 \cdots p_{\frac{4^t-1}{3}} n - \frac{4^t-1}{3}}{8}\right) \equiv 0 \pmod{2}.$$

Moreover, Boylan [Boy02, Theorem 1.2] also proved that for all positive integers n and for any odd prime p such that $\gcd(n, p) = 1$,

$$c_{2^t}\left(\frac{p^{2^{2t-j}-1} n - \frac{4^t-1}{3}}{8}\right) \equiv 0 \pmod{2},$$

where $j = 1, 2, 3,$ or 4 depending on p and t. Chen [Che13, Theorem 3] improved Boylan's congruence equation: for a positive integer t and distinct odd primes $p_1, p_2, \ldots, p_{2^{t-1}}$ with $\gcd(n, \prod p_i) = 1$,

$$c_{2^t}\left(\frac{p_1 p_2 \cdots p_{2^{t-1}}\, n - \frac{4^t-1}{3}}{8}\right) \equiv 0 \pmod{2}.$$

4. Simultaneous cores

For positive integers $t_1, t_2, \ldots, t_p$, a partition is called a $(t_1, t_2, \ldots, t_p)$-*core partition* if it is a t_1-core, a t_2-core, ..., and a t_p-core simultaneously. Since $(k(t-1)+1, \ldots, 2(t-1)+1, (t-1)+1, 1)$ is a t-core partition for all positive integers k, the number of t-core partitions is infinite. However, it is known that, for $\gcd(t_1, t_2, \ldots, t_p) = 1$, the number of $(t_1, t_2, \ldots, t_p)$-core partitions is finite. Otherwise, the number of $(t_1, t_2, \ldots, t_p)$-cores is infinite since $\gcd(t_1, t_2, \ldots, t_p)$-cores are also $(t_1, t_2, \ldots, t_p)$-cores.

In Section 4.A., we list results regarding simultaneous (s, t)-core partitions for two coprime integers s and t, and Section 4.B. contains more general results on simultaneous core partitions.

4.A. Results on (s,t)-cores

When $\gcd(s,t)=1$, Anderson proved that the number of (s,t)-core partitions is finite and it is a rational Catalan number.

Theorem 4.1. *[And02, Theorem 1] For two coprime positive integers s and t, the number of (s,t)-core partitions is*

$$\mathrm{Cat}_{s,t}=\frac{1}{s+t}\binom{s+t}{s}.$$

Since the number of (s,t)-core partitions is finite, we now restrict the set of (s,t)-core partitions by certain conditions and count them. It is natural to consider how many (s,t)-core partitions with additional restrictions are. The first restriction is allowing only "distinct" parts in the partition. A partition is called distinct if all the parts are different. Amdeberhan [Amd12, Conjecture 11.10] proposed two interesting conjectures. The first conjecture is that the number of $(t,t+1)$-core partitions into distinct parts is F_{t+1}, the $(t+1)$st Fibonacci number. The second one is that, for an odd number t, the number of $(t,t+2)$-core partitions into distinct parts is 2^{t-1}. Straub [Str16, Theorem 2.1] and Xiong [Xio18, Theorem 1.2] independently proved the first conjecture. Moreover, Straub [Str16, Theorem 1.4] found a recurrence relation of the number $E_d^-(t)$ of $(t,dt-1)$-core partitions into distinct parts:

$$E_d^-(1)=1, E_d^-(2)=d, \text{ and, for } t\geq 3, E_d^-(t)=E_d^-(t-1)+dE_d^-(t-2).$$

Also, Nath and Sellers [NaSe17, Theorem 15] gave the result on the number $E_d^+(t)$ of $(t,dt+1)$-core partitions into distinct parts:

$$E_d^+(1)=1, E_d^+(2)=d+1, \text{ and, for } t\geq 3, E_d^+(t)=E_d^+(t-1)+dE_d^+(t-2).$$

The second conjecture was proved in three different ways by Yan, Qin, Jin, and Zhou [YQJZ17, Conjecture 1.1], by Zaleski and Zeilberger [ZaZe17, Theorem 0], and by Baek, Nam, and Yu [BNY18, Corollary 1.2]. To prove the conjecture, Yan, Qin, Jin, and Zhou manipulated formulas involving binomials and Catalan numbers by combining bijection arguments, and Zaleski and Zeilberger used their symbolic-computational algorithms to compute the number of $(t,t+2)$-cores into distinct parts and the first 22 moments of the distribution of sizes of the $(t,t+2)$-cores into distinct parts. Inspired by Yan, Qin, Jin, and Zhou's bijective arguments, Baek, Nam, and Yu found the first direct bijective proof of the result.

We say that a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ has d-distinct parts if $\lambda_i-\lambda_{i+1}\geq d$ for each i. For any positive integer d, let $N_{d,r}(t)$ be the number of $(t,t+r)$-core partitions with d-distinct parts. Sahin [Sah18, Theorem 2.2] showed the following recurrence relation:

$$N_{d,1}(t)=\begin{cases} t & \text{if } 1\leq t\leq d+1,\\ N_{d,1}(t-1)+N_{d,1}(t-d-1) & \text{if } t\geq d+2.\end{cases}$$

Kravitz [Kra19, Theorem 2.9] found a general version of Sahin's result for any positive integers $r\leq d$ and any positive integer t. We have

$$N_{d,r}(t)=\sum_{i=0}^{\lceil\frac{t-1}{d+1}\rceil}\binom{t+d-di-1}{i}+(r-1)\sum_{i=0}^{\lceil\frac{t-2d-1}{d+1}\rceil}\binom{t-d-di-1}{i}.$$

Now we count the number of self-conjugate core partitions. Ford, Mai, and Sze [FMS09, Theorem 1] proved that, for two coprime positive integers s and t, the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor\frac{s}{2}\rfloor+\lfloor\frac{t}{2}\rfloor}{\lfloor\frac{s}{2}\rfloor}.$$

In addition to counting the number of simultaneous core partitions, mathematicians observed the size of simultaneous core partitions. The study of the size of core partitions was initiated by Armstrong, Hanusa, and Jones. They conjectured that,

Conjecture 4.2. *[AHJ14, Conjecture 2.6] For two coprime positive integers s and t, the average size of an (s,t)-core partition is*

$$\frac{1}{24}(s+t+1)(s-1)(t-1).$$

The average size of a self-conjugate (s,t)-core partition is also the same as above.

In the aspect of an (s,t)-core partition, this conjecture was partially proved by Stanley and Zanello [StZa15, Theorem 2.3] when $t = s+1$. Aggarwal [Agg15, Theorem 1.2] generalized the proof to the case when $t \equiv 1 \pmod{s}$. Later, Johnson [Joh18, Theorem 7] gave the full proof using Ehrhart theory. The self-conjugate part of the conjecture was proved by Chen, Huang, and Wang [CHW16, Theorem 2.1].

Next topic is about the largest size of simultaneous core partitions. Since there are only finite number of simultaneous core partitions, one may ask how large the size could be. Olsson and Stanton [OlSt07, Theorem 4.1] showed that, when s and t are coprime, there is a unique largest (s,t)-core partition of size

$$\frac{1}{24}(s^2-1)(t^2-1),$$

which turns out to be self-conjugate. Adding a condition to be distinct, Yan, Qin, Jin, and Zhou [YQJZ17, Conjecture 1.2] evaluated the largest size of a $(t,t+2)$-core partition into distinct parts, which is

$$\frac{1}{384}(t^2-1)(t+3)(5t+17).$$

Nam and Yu [NaYu21, Theorem 1.1] considered the parity of the parts of a partition. They proved that the largest size of a $(t,t+1)$-core partition which all of its parts are odd is

$$\begin{cases} \frac{1}{96}(t^4+4t^3+26t^2+44t-75) & \text{if } t \text{ is odd,}\\ \frac{1}{96}(t^4+4t^3+20t^2+32t-96) & \text{if } t \text{ is even,}\end{cases}$$

and the largest size of a $(t,t+1)$-core partition which all of its parts are even is

$$\begin{cases} \frac{1}{96}(t^4+4t^3+2t^2-4t-3) & \text{if } t \text{ is odd,}\\ \frac{1}{96}(t^4+4t^3-4t^2-16t) & \text{if } t \text{ is even.}\end{cases}$$

4.B. More on simultaneous cores

After the various results on (s,t)-core partitions, many people have examined simultaneous core partitions with at least three cores. However, it seems to be infeasible unless there is a pattern on cores. Therefore, researchers first dealt with simultaneous core partitions whose cores form an arithmetic progression.

Amdeberhan [Amd12, Conjectures 11.1–11.3] posed three conjectures about $(t,t+1,t+2)$-core partitions which were proved by Yang, Zhong, and Zhou [YZZ15, Conjectures 1.1–1.3]. By considering the order ideals in a poset structure, they counted the number, the largest size, and the sum of sizes of $(t,t+1,t+2)$-core partitions. The number of $(t,t+1,t+2)$-core partitions is $\sum_{k\geq 0}\binom{t}{2k}C_k$, where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number, the largest size of a $(t,t+1,t+2)$-core partition is

$$\begin{cases} n\binom{n+1}{3} & \text{if } t = 2n-1,\\ (n+1)\binom{n+1}{3}+\binom{n+2}{3} & \text{if } t = 2n,\end{cases}$$

and the sum of the sizes of all $(t, t+1, t+2)$-core partitions equals

$$\sum_{j=0}^{t-2} \binom{j+3}{3} \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2i} C_i.$$

The first two results have been generalized individually by Amdeberhan and Leven and by Xiong. Amdeberhan and Leven [AmLe15, Section 4] gave a path interpretation and a recurrence formula for $(t, t+1, \ldots, t+p)$-core partitions using the definition of generalized Dyck path. For more details, see [AmLe15].

On the other hand, Xiong [Xio16, Theorem 1.1] generalized a result of Yang, Zhong, and Zhou. Let t and p be positive integers such that $t = pd + n$, where $1 \le d \le p$ and $n \ge 0$. Then, the largest size of a $(t, t+1, \ldots, t+p)$-core partition is

$$\max\left\{ \binom{n+2}{2} \left\lfloor \frac{d}{2} \right\rfloor \left(d - \left\lfloor \frac{d}{2} \right\rfloor \right) + \binom{n+2}{3}(p^2 n + pd - p^2) - 3\binom{n+2}{4} p^2, \right.$$
$$\left. \binom{n+1}{2} \left(p - \left\lfloor \frac{p-d}{2} \right\rfloor \right) \left(d + \left\lfloor \frac{p-d}{2} \right\rfloor \right) + \binom{n+1}{3}(p^2 n + pd - p^2) - 3\binom{n+1}{4} p^2 \right\}.$$

Amdeberhan [Amd12, Conjecture 11.5] also conjectured that the number of $(t, t+d, t+2d)$-core partitions is

$$\frac{1}{t+d} \sum_{i \ge 0} \binom{t+d}{i, i+d, t-2i},$$

for coprime positive integers t and d. It was proved by V. Wang [Wan16, Theorem 1.6].

Baek, Nam, and Yu [BNY19, Corollary 5.6 and Theorem 5.7] found the number of $(t, t+d, t+2d)$-core partitions alternatively and gave an explicit formula for the number of $(t, t+d, t+2d, t+3d)$-core partitions, for relatively prime positive integers t and d, which is

$$\frac{1}{t+d} \sum_{k=0}^{\lfloor t/2 \rfloor} \left\{ \binom{t+d-k}{k} + \binom{t+d-k-1}{k-1} \right\} \binom{t+d-k}{s-2k}.$$

The overall generalization of arithmetic progression cases of core partitions was given by Cho, Huh, and Sohn.

Theorem 4.3. *[CHS20a, Theorem 1.5] Let t and d be relatively prime positive integers. For $p \ge 2$, the number of $(t, t+d, \ldots, t+pd)$-cores is*

$$\frac{1}{t+d}\binom{t+d}{d} + \sum_{k=1}^{\lfloor \frac{t}{2} \rfloor} \sum_{\ell=0}^{r} \frac{1}{k+d} \binom{k+d}{k-\ell} \binom{k-1}{\ell} \binom{t+d-\ell(p-2)-1}{2k+d-1},$$

where $r = \min(k-1,\ \lfloor (s-2k)/(p-2) \rfloor)$.

From now on, we deal with results on self-conjugate simultaneous core partitions. Cho, Huh, and Sohn [CHS21, Theorem 4] showed that, for a positive integer t, the number of self-conjugate $(t, t+1, t+2)$-core partitions is

$$\sum_{i \ge 0} \binom{\lfloor \frac{t}{2} \rfloor}{i} \binom{i}{\lfloor \frac{i}{2} \rfloor}.$$

Baek, Nam, and Yu [BNY19, Theorem 3.3] found the largest size of a self-conjugate $(t,t+1,t+2)$-core partition, which is

$$\begin{cases} \frac{n(2n+1)(4n^2+2n+1)}{3} & \text{if } t=4n, \\ \frac{n^2(8n^2-6n+1)}{3} & \text{if } t=4n-1, \\ \frac{n(2n-1)(4n^2-2n+1)}{3} & \text{if } t=4n-2, \\ \frac{(n-1)(2n-1)(4n^2-5n+3)}{3} & \text{if } t=4n-3. \end{cases}$$

Moreover, they proved that there is a unique self-conjugate $(t,t+1,t+2)$-core partition having the largest size.

Yan, Yu, and Zhou [YYZ20, Theorems 2.14, 2.19, and 2.22] gave a bijection between the set of $(t,t+1,\dots,t+p)$-core partitions and that of symmetric (t,p)-generalized Dyck paths.

Cho, Huh, and Sohn [CHS20a, Theorem 3.9] showed that the number of self-conjugate $(t,t+1,\dots,t+p)$-core partitions is

$$1+\sum_{k=1}^{\lfloor t/2\rfloor}\sum_{\ell=0}^{r}\binom{\lfloor\frac{k-1}{2}\rfloor}{\lfloor\frac{\ell}{2}\rfloor}\binom{\lfloor\frac{k}{2}\rfloor}{\lfloor\frac{\ell+1}{2}\rfloor}\binom{\lfloor\frac{t-\ell(p-2)}{2}\rfloor}{k},$$

where $p\geq 2$ and $r=\min(k-1,\lfloor(t-2k)/(p-2)\rfloor)$.

Recently, Cho and Huh [ChHu, Theorems 3.3 and 3.4] enumerated the number of self-conjugate $(t,t+d,t+2d)$-core partitions and the number of self-conjugate $(t,t+d,t+2d,t+3d)$-core partitions.

Let t and d be relatively prime positive integers. The number of self-conjugate $(t,t+d,t+2d)$-core partitions is given by

$$\begin{cases} \displaystyle\sum_{i=0}^{\lfloor\frac{t}{4}\rfloor}\binom{\frac{t+d-1}{2}}{i,\ \frac{d}{2}+i,\ \frac{t-1}{2}-2i}, & \text{if } d \text{ is even,} \\ \displaystyle\sum_{i=0}^{\lfloor\frac{t}{2}\rfloor}\binom{\lfloor\frac{t+d-1}{2}\rfloor}{\lfloor\frac{i}{2}\rfloor,\ \lfloor\frac{d+i}{2}\rfloor,\ \lfloor\frac{t}{2}\rfloor-i}, & \text{if } d \text{ is odd,} \end{cases}$$

and the number of self-conjugate $(t,t+d,t+2d,t+3d)$-core partitions is given by

$$\begin{cases} \displaystyle\sum_{i=0}^{\lfloor\frac{t}{4}\rfloor}\binom{\frac{t+d-1}{2}-i}{\frac{t-1}{2}-2i}\binom{\frac{t+d-1}{2}-i}{i}, & \text{if } d \text{ is even,} \\ \displaystyle\sum_{i=0}^{\lfloor\frac{t}{2}\rfloor}\binom{\lfloor\frac{t+d-1}{2}\rfloor-\lfloor\frac{i}{2}\rfloor}{\lfloor\frac{t}{2}\rfloor-i}\binom{\lfloor\frac{t+d}{2}\rfloor-\lfloor\frac{i+1}{2}\rfloor}{\lfloor\frac{i}{2}\rfloor}, & \text{if } d \text{ is odd.} \end{cases}$$

Only a few results on simultaneous core partitions with at least three cores are known when those cores do not form an arithmetic progression. The following is a result proved by Baek, Nam, and Yu.

Theorem 4.4. *[BNY19, Theorem 5.4] Let $s,t_0,t_1,\dots,t_n$ be positive integers, where none of t_i is a multiple of s. Suppose that s and t_0 are relatively prime. For $1\leq i\leq n$, let l_i be such that $1\leq l_i\leq s-1$ and $s\mid t_0l_i+t_i$. Then, the number of $(s,t_0,t_1,\dots,t_n)$-core partitions is*

$$\frac{1}{s}\left|\left\{(z_0,\dots,z_{s-1})\in\mathbb{N}^s:\sum_{m=0}^{s-1}z_m=t_0 \text{ and } \sum_{m=j}^{j+l_i-1}z_m\leq\frac{t_0l_i+t_i}{s} \text{ for all } i,j\right\}\right|.$$

5. Applications of t-cores

In this section, we describe roles of t-core partitions in the theory of partitions and in number theory. We start with a relation between self-conjugate core partitions and ordinary core partitions. In Section 5.2, we introduce a 5-core crank which combinatorially explains Ramanujan's partition congruence modulo 5. In the last subsection, we explain how class numbers are related with 4-cores and self-conjugate 7-cores.

5.A. t-cores and self-conjugate t-cores

In this subsection, we give an identity between self-conjugate $2t$-core partitions and ordinary t-core partitions. First, we recall generating functions of $c_t(n)$ and $sc_{2t}(n)$.

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{5.6}$$

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = (-q;q^2)_\infty (q^{4t};q^{4t})_\infty^t. \tag{5.7}$$

By combining (5.6), (5.7), and the Gauss identity, we have the following identity.

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = \left(\sum_{n=0}^{\infty} c_t(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right) \quad \text{for } |q|<1. \tag{5.8}$$

Cho, Huh, and Sohn [CHS20b, Corollary 1.4] gave a combinatorial proof of (5.8) by using the bijection of Wright for the Jacobi triple product identity (see [Pak06, Wri65]). They also gave a generalization of (5.8).

Theorem 5.1. *[CHS20b, Theorem 1.3] Let $c_{(t_1,\dots,t_p)}(n)$ be the number of $(t_1,\dots,t_p)$-core partitions of n and $sc_{(t_1,\dots,t_p)}(n)$ be the number of self-conjugate $(t_1,\dots,t_p)$-core partitions of n. For $|q|<1$,*

$$\sum_{n=0}^{\infty} sc_{(2t_1,\dots,2t_p)}(n)q^n = \left(\sum_{n=0}^{\infty} c_{(t_1,\dots,t_p)}(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right).$$

5.B. t-core crank

To explain Ramanujan's striking partition congruences, several partition statistics have been proposed. In number theoretic aspect, Dyson's rank [Dys44] and Andrews-Garvan's crank [AnGa88, Gar88] are very interesting since Dyson's rank generating function is deeply related with mock modular forms [BFOR17, Zag10] and Andrews-Garvan's crank generating function can be understood via Jacobi forms [EiZa85, Mah05]. In combinatorial aspect, Garvan, Kim, and Stanton's t-core crank is arguably most interesting. By constructing explicit statistics on t-cores of ordinary partitions, they gave a unifying approach to explain Ramanujan's three congruences modulo 5, 7, and 11, and the congruence $p(25n+24) \equiv 0 \pmod{25}$. For the sake of brevity, here we only give the crank for the congruence $p(5n+4) \equiv 0 \pmod 5$.

Garvan, Kim, and Stanton first constructed a bijection implying

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q^t;q^t)_\infty} \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\|\vec{n}\|^2+\vec{b}\cdot\vec{n}},$$

where $\vec{b} = (0, 1, 2, \dots, t-1)$. One can find a generating function for $p(5n+4)$ by gathering the terms $\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} \equiv 4 \pmod 5$. To see an automorphism group for the quadratic form in the exponent more naturally, they changed the variable properly and proved that [GKS90, Theorem 1],

$$\sum_{n\geq 0} p(5n+4)q^{n+1} = \frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1\\ \vec{\alpha}\in\mathbb{Z}^5}} q^{Q(\vec{\alpha})},$$

where $Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_0)$. One can immediately see that the dihedral group D_5 is the automorphism group for $Q(\vec{\alpha})$ and there is no fixed point under the rotation. This gives a combinatorial explanation how one can divide partitions of $5n+4$ into five equinumerous classes. Surprisingly, the same idea works for $p(7n+5)$ and $p(11n+6)$, and gives a generating function in a uniform fashion (see [GKS90, Theorem 1]). Moreover, this idea can be extended to $p(25n+24) \equiv 0 \pmod{25}$ [GKS90, Theorem 6] and $p(49n+47) \equiv 0 \pmod{49}$ [Gar01]. While this is combinatorially beautiful, it is not immediate how to calculate crank from the part sizes of the partition directly. Thus, they derived a crank, which in this paper it refers as a t-core crank to avoid the ambiguity, using the part sizes as follows [GKS90, Theorem 3].

Theorem 5.2. *For a partition λ, let*

$$5\text{--core crank}(\lambda) := \sum_{i=1}^{\ell} ((\lambda_i - i - 2)^2 - (i-3)^2),$$

where ℓ is the number of parts. Then, there are the same number of partitions of $5n+4$ with the 5-core crank congruent to i modulo 5 for all integers $0 \leq i \leq 4$.

Here we give a table to show how Dyson's ranks, Andrews-Garvan's cranks, and t-core cranks divide 5 partitions of 4 into five classes.

Partitions of 4	rank (mod 5)	crank (mod 5)	5-core crank (mod 5)
4	3	4	2
3+1	1	0	4
2+2	0	2	0
2+1+1	4	3	1
1+1+1+1	2	1	3

Table 1: Partitions of 4 and their rank, crank, and 5-core crank.

5.C. Class numbers

In the previous subsection, we have seen that there is a natural connection between core partitions and quadratic forms. This connection is particularly interesting when a generating function for core partitions is essentially a modular form of weight 3/2. Let $h(D)$ denote the discriminant D class number, i.e., the order of the class group of the discriminant D binary quadratic forms.[1] For more details on quadratic forms and its class numbers, see [IrRo90, Jon50, O'Me00]. Ono and Sze [OnSz97, Theorem 2] proved that

$$c_4(n) = \frac{1}{2}h(-32n-20)$$

[1]Some authors use $H(D)$ or $H(|D|)$ instead of $h(D)$.

if $8n+5$ is square-free. The proof follows from the fact that a generating function for 4-cores can be expressed as a product of three theta functions, which leads that $c_4(n)$ is essentially the number of representations of $8n+5$ as $x^2+2y^2+2z^2$ with $x,y,z>0$ since

$$\begin{aligned}\sum_{n\geq 0} c_4(n)q^{8n+5} &= q^5\frac{(q^{32};q^{32})_\infty^4}{(q^8;q^8)_\infty} \\ &= q^5\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\frac{(q^{32};q^{32})_\infty^4}{(q^{16};q^{16})_\infty^2} \\ &= \sum_{k\geq 0} q^{(2k+1)^2}\left(\sum_{k\geq 0} q^{2(2k+1)^2}\right)^2 \\ &= \sum_{n\geq 0} \left|\{(x,y,z)\in\mathbb{N}^3 : 8n+5 = x^2+2y^2+2z^2\}\right| q^{8n+5}.\end{aligned}$$

From the genus theory of quadratic forms and arithmetic properties of $c_4(n)$, one can find that $|\{(x,y,z)\in\mathbb{N}^3 : 8n+5 = x^2+2y^2+2z^2\}| = \frac{1}{2}h(-32n-20)$. For instance, Ono and Sze [OnSz97, Corollary 3] proved the congruence

$$c_4(n)\equiv 0 \pmod{2^k}$$

provided that $8n+5$ is a square-free integer with k prime divisors. More recently, Ono and Raji [OnRa21, Theorem 1] obtained a connection between self-conjugate 7-core partitions and class numbers, namely, for a positive odd integer $n\not\equiv 5 \pmod 7$,

$$sc_7(n) = \begin{cases} \frac{1}{4}h(-28n-56) & \text{if } n\equiv 1 \pmod 4, \\ \frac{1}{2}h(-7n-14) & \text{if } n\equiv 3 \pmod 8, \\ 0 & \text{if } n\equiv 7 \pmod 8. \end{cases}$$

This follows again from the fact that a generating function, namely, $S(q)=\sum_{n\geq 0} sc_7(n)q^{n+2} = \frac{\eta^2(2z)\eta(14z)\eta(28z)}{\eta(4z)\eta(z)}$ is a holomorphic modular form of weight $3/2$ on $\Gamma_0(28)$ with the Kronecker character $(7/n)$.

Bringmann, Kane, and Males [BKM21, Equation (1.1)] observed that the above two expressions involving class numbers imply that

$$2sc_7(8n+1) = c_4(7n+2) \tag{5.9}$$

provided that $n\not\equiv 4 \pmod 7$ and $56n+21$ is square-free. Based on this observation, they tried to find more linear relations between t-core partitions and self-conjugate $(2t-1)$-core partitions like (5.9), but they found that there is no such a linear relation for $t=2$, 3, or 5 and based on this, they conjectured that there is no linear relation like (5.9) for $t\neq 4$.

6. Concluding Remarks

Even though we have focused on the roles of t-cores in the theory of partitions and its number theoretic aspects, it is infeasible to introduce all interesting results in this short survey. Here we mention a short list of interesting results. There have been an extensive study on the role of hook lengths. For example, one may see the papers [BeHan09, Han10, HaJi11, HaOn11, HaXi17] of Han and his collaborators. More recently, Bringmann, Ono, and Wagner [BOW20] found a new connection between the modularity and the hook length. There are also many interesting results on the simultaneous core partitions with additional conditions. For example, Zhou and Yan [ZY17] showed that the number of $(t,t+1)$-core partitions with parts that are multiples of p is related with the Fuss-Catalan number, and Huang and Wang [HuWa18] found the number of $(t, t+1)$-core partitions with m corners, which is the Narayana number $N(t, m+1) = \frac{1}{t}\binom{t}{m+1}\binom{t}{m}$. There are also combinatorial interpretations using paths for the number of simultaneous core partitions. See [AmLe15, And02, AHJ14, ChHu, CHS20a, FMS09, YYZ20] for more details. It is also possible to think t-cores in the random partition [AySi19].

Another interesting object that is related to core partitions is a numerical set. One can see that there is a bijection between the set of t-core partitions and the set of numerical sets. Moreover, there is a bijection between the set of $(t_1, t_2, \ldots, t_k)$-core partitions and the set of integer points in a certain polytope. For more details, see [CHK17].

Surely there are many further interesting results on t-cores we haven't mentioned. This is not because the results are not interesting but because the authors' short scope.

While there have been much progresses on the study of t-cores, there are still many open questions and identities waiting for further investigations. We have already introduced several open questions like Stanton's conjecture, and Hanusa and Nath's conjecture. Most linear relations among t-cores and self-conjugate t-cores are still combinatorially mysterious.

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