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Partition Identities for Two-Color Partitions

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In memory of Srinivasa Ramanujan

Abstract. Three new partition identities are found for two-color partitions. The first relates to ordinary partitions into parts not divisible by 4, the second to basis partitions, and the third to partitions with distinct parts. The surprise of the strangeness of this trio becomes clear in the proof.

Keywords. Partitions, Two-Color Partitions, Rogers-Ramanujan identities.

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1. Introduction.

The first paper on this topic [And87] appeared in 1987 to celebrate the $100^{\underline{\text{th}}}$ anniversary of Ramanujan's birth. It seems fitting to continue the study in this volume marking the $100^{\underline{\text{th}}}$ anniversary of his death.

Two-color partitions are formed from two copies of the integers. We denote red integers with the subscript r, and green integers with the subscript g. Thus there are five two-color partitions of 2, namely $2_r, 2_g, 1_r + 1_r, 1_r + 1_g, 1_g + 1_g$. We shall say that two parts of a partition are distinct if they are of different colors or different numerical values or both. We shall say that two parts are numerically distinct if they have different numerical values.

The two partition identities proved in [And87] are these:

Theorem 1.1. *Let $L_1(n)$ denote the number of two-color partitions of n into numerically distinct parts wherein two summands cannot be consecutive integers of the same color. Let $L_2(n)$ denote the number of two-color partitions of n in which all parts are odd and no green parts are repeated. Then for all n,*

$$L_1(n) = L_2(n).$$

Theorem 1.2. *Let $G(n)$ denote the number of two-color partitions of n in which the largest part is red, the parts are numerically distinct, and each green part is at least 2 smaller than the next largest part. Let $S_1(n)$ denote the number of ordinary partitions of n into parts that are either odd or congruent to $\pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}$. Let $S_2(n)$ denote the number of ordinary partitions of n into parts that are either odd or congruent to $\pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}$. Then for all $n \geq 1$,*

$$G(n) = S_1(n) = S_2(n-1).$$

As Jeremy Lovejoy remarks in [Lov03, p. 395], $L_2(n)$ is essentially the number of overpartitions of n into odd parts. While partition identities for overpartitions have flourished, there has not been any follow-up revealing further partition identities for two-color partitions.

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The purpose of this paper is to give three further results of this nature. We define $\mathcal{L}_d(n)$ to be the number of two-color partitions into numerically distinct parts with the added condition that red parts are at least d larger than the next largest part and green parts are at least $d+1$ larger than the next largest part, and furthermore with no green part 1_g or $(d-1)_g$.

Theorem 1.3. *$\mathcal{L}_1(n)$ equals the number of partitions of n in which no part is divisible by 4.*

Theorem 1.4. *$\mathcal{L}_2(n)$ equals the number of basis partitions of n.*

Theorem 1.4 was proved in [And15, sec. 4] by a slightly different approach. It should be noted that in [And15, p. 67] five lines following equation (4.3) the word "smaller" should be "larger." The theorem in [And15] is stated in terms of the overpartitions, but the mapping of red parts to nonoverlined parts and overlined parts to green parts yields the current Theorem 1.4.

Theorem 1.5. *$\mathcal{L}_3(n)$ equals the number of partitions of n into odd parts (i.e. no part is divisible by 2).*

The next three sections will provide proofs of Theorems 1.3-1.5. An alternative proof of Theorem 1.5 will be given in section 5. The final section will be devoted to a conjecture that arises naturally from a corollary of Theorems 1.3-1.5.

2. Proof of Theorem 1.2.

Proof. Let $S_1(N)$ denote the generating function for partitions of the type enumerated by $\mathcal{L}_1(N)$ with the added condition that each part is $< N$.

Then

$$S_1(N) = \begin{cases} 0 & \text{if } N<0 \\ 1 & \text{if } N=0 \text{ or } 1 \\ 1+q & \text{if } N=2 \\ S_1(N-1)+q^{N-1}(S_1(N-1)+S_1(N-2)) & \text{otherwise.} \end{cases} \tag{2.1}$$

The first three lines in (2.1) are immediate. The last line follows by dividing the two-color partitions enumerated by $S_1(N)$ into three classes: (i) those in which no part is numerically equal to $N-1$, (ii) those in which $(N-1)_r$ is a part, (iii) those in which $(N-1)_g$ is a part.

Now define

$$\mathcal{S}_1(x) := \sum_{N\geq 0} S_1(N)x^N.$$

Hence

$$\begin{aligned} \mathcal{S}_1(x) &= 1+x+(1+q)x^2+\sum_{N\geq 3}(S_1(N-1)(1+q^{N-1})+q^{N-1}S_1(N-2)x^N) \\ &= 1+x+(1+q)x^2+\sum_{N\geq 2} S_1(N)(1+q^N)+\sum_{N\geq 1} q^{N+1}S_1(N)x^{N+2} \\ &= 1+x+(1+q)x^2+(x\,\mathcal{S}_1(x)-1-x)+x(\mathcal{S}_1(xq)-1-xq)+x^2q(\mathcal{S}_1(xq)-1) \\ &= 1-x-qx^2+x\,\mathcal{S}_1(x)+x(1+xq)\,\mathcal{S}_1(xq). \end{aligned} \tag{2.2}$$

Consequently

$$\mathcal{S}_1(x) = \frac{1-x-qx^2}{1-x}+\frac{x(1+xq)}{1-x}\,\mathcal{S}_1(xq). \tag{2.3}$$

Iterating (2.3), we obtain

$$\mathcal{S}_1(x) = \sum_{n\geq 0}\frac{(1-xq^n-x^2q^{2n+1})x^nq^{\binom{n}{2}}(-xq)_n}{(x)_{n+1}}, \tag{2.4}$$

where $(A)_N = (A;q)_N = \prod_{j=0}^{N-1}(1-Aq^j)$. Now Abel's lemma [And94, p. 190, Th. 14-7] reveals that

$$\begin{aligned}
\sum_{n\geq 1}\mathcal{L}_1(n)q^n &= \lim_{x\to 1^-}(1-x)\,\mathcal{S}_1(x)\\
&= \sum_{N\geq 0}\frac{(1-q^N-q^{2N+1})q^{\binom{N}{2}}(-q)_N}{(q)_N}\\
&= \sum_{N\geq 1}\frac{q^{\binom{N}{2}}(-q)_N}{(q)_{N-1}} - \sum_{N\geq 0}\frac{q^{\binom{N}{2}+2N+1}(-q)_N}{(q)_N}\\
&= \sum_{N\geq 0}\frac{q^{\binom{N+1}{2}}(-q)_{N+1}}{(q)_N} - \sum_{N\geq 0}\frac{q^{\binom{N}{2}+2N+1}(-q)_N}{(q)_N}\\
&= \sum_{N\geq 0}\frac{q^{\binom{N+1}{2}}(-q)_N(1+q^{N+1}-q^{N+1})}{(q)_N}\\
&= \sum_{N\geq 0}\frac{q^{\binom{N+1}{2}}(-q)_N}{(q)_N}\\
&= (-q^2;q^2)_\infty(-q)_\infty \text{ (by [And76, p. 21, Cor. 2.7])}\\
&= \frac{(q^4;q^4)_\infty}{(q)_\infty}.
\end{aligned}\tag{2.5}$$

Since that latter product is the generating function for partitions in which no part is divisible by 4, we see that the extremes of (2.5) establish Theorem 1.3.

3. Proof of Theorem 1.4.

Proof. Basis partitions were originally introduced by Hansraj Gupta. Their definition is rather complicated, but all that is required here is the generating function due to Nolan, Savage, and Wilf [Nol98]:

$$\sum_{n\geq 0}\frac{(-q)_n q^{n^2}}{(q)_n}.\tag{3.6}$$

Let $S_2(N)$ denote the generating function for partitions of the type enumerated by $\mathcal{L}_2(N)$ with the added condition that each part is $< N$. Then

$$S_2(N) = \begin{cases} 0 & \text{if } N<0\\ 1 & \text{if } N=0 \text{ or } 1\\ S_2(N-1)+q^{N-1}(S_2(N-2)+S_2(N-3)) & \text{otherwise}\end{cases}.\tag{3.7}$$

The same reasoning as in the proof of Theorem 1.3 establishes (3.7).

Now define

$$\mathcal{S}_2(x) := \sum_{N\geq 0} S_2(N)x^N.$$

Then

$$\begin{aligned}\mathcal{S}_2(x) &= 1 + x + \sum_{N\geq 2}\left(S_2(N-1) + q^{N-1}(S_2(N-2) + S_2(N-3))\right)x^N\\ &= 1 + x + \sum_{N\geq 1} S_2(N)x^{N+1} + \sum_{N\geq 0} q^{N+1}S_2(N)x^{N+2} + \sum_{N\geq -1} q^{N+2}S_2(N)x^{N+3}\\ &= 1 + x + x(\mathcal{S}_2(x) - 1) + x^2q\,\mathcal{S}_2(xq) + x^3q^2\,\mathcal{S}_2(xq)\\ &= 1 + x\,\mathcal{S}_2(x) + x^2q(1+xq)\,\mathcal{S}_2(xq).\end{aligned}$$

Consequently,

$$\mathcal{S}_2(x) = \frac{1}{1-x} + \frac{x^2q(1+xq)}{1-x}\,\mathcal{S}_2(xq). \tag{3.8}$$

Iterating (3.8), we see that

$$\mathcal{S}_2(x) = \sum_{n\geq 0}\frac{x^{2n}q^{n^2}(-xq)_n}{(x)_{n+1}}.$$

Again by Abel's lemma [And94, p. 190, Th. 14-7],

$$\sum_{n\geq 1}\mathcal{L}_2(n)q^n = \lim_{x\to 1^-}(1-x)\,\mathcal{S}_2(x) = \sum_{n\geq 0}\frac{q^{n^2}(-q)_n}{(q)_n},$$

and Theorem 1.4 now follows from (3.6).

4. Proof of Theorem 1.5.

Proof. As is by now familiar and expected, we let $S_3(N)$ denote the generating function for partitions of the type enumerated by $\mathcal{L}_3(N)$ with the added condition that each part is $< N$. Then

$$S_3(N) = \begin{cases} 0 & \text{if } N < 0\\ 1 & \text{if } N = 0 \text{ or } 1\\ 1+q & \text{if } N = 2\\ 1+q+q^2 & \text{if } N = 3\\ S_3(N-1) + q^{N-1}(S_3(N-3) + S_3(N-4)) & \text{otherwise.}\end{cases} \tag{4.9}$$

Note that the initial condition $S_3(3) = 1 + q + q^2$ follows from the fact that 2_g is excluded from the definition of $\mathcal{L}_a(n)$ when $n = 3$ by "no green part 1_g or $(d-1)_g$." The remaining justification for (4.9) is precisely like that of the two previous theorems.

Hence, if we define

$$\mathcal{S}_3(x) = \sum_{N\geq 00} S_3(N)x^N,$$

Then

$$\begin{aligned}\mathcal{S}_3(x) &= 1 + x + x^2(1+q) + x^3(1+q+q^2)\\ &\quad + \sum_{N\geq 4} x^N(S_3(N-1) + q^{N-1}(S_3(N-3) + S_3(N-4)))\\ &= 1 + x + x^2(1+q) + x^3(1+q+q^2) + \sum_{N\geq 3} x^{N+1}S_3(N)\\ &\quad + \sum_{N\geq 1} x^{N+3}q^{N+2}S_3(N) + \sum_{N\geq 0} x^{N+4}q^{N+3}S_3(N)\\ &= 1 + x + x^2(1+q) + x^3(1+q+q^2)\\ &\quad + x(\mathcal{S}_3(x) - 1 - x - x^2(1+q)) + x^3(\mathcal{S}(xq) - 1) + x^4q^3\,\mathcal{S}_3(xq)\end{aligned}$$

Consequently,

$$\mathcal{S}_3(x) = \frac{1+x^2q}{1-x} + \frac{x^3q^2(1+xq)}{1-x}\,\mathcal{S}_3(xq). \tag{4.10}$$

Iterating (4.10), we see that

$$\mathcal{S}_3(x) = \sum_{n\geq 0} \frac{x^{3n}(1+x^2q^{2n+1})(-xq)_n q^{n(3n+1)/2}}{(x)_{n+1}}.$$

Finally, applying Abel's lemma [And94, p. 190, Th. 14-7],

$$\begin{aligned}\sum_{n\geq 0} \mathcal{L}_3(n)q^n &= \lim_{x\to 1^-}(1-x)\,\mathcal{S}_3(x) = \sum_{n\geq 0}\frac{(1+q^{2n+1})(-q)_n q^{n(3n+1)/2}}{(q)_n}\\ &= (-q)_\infty \text{ (by [And76, p. 140, Th. 9.2,} x=-1\text{])}\\ &= \frac{1}{(q;q^2)_\infty} \text{ (by [And76, p. 5, eq. (1.2.5)])}\end{aligned} \tag{4.11}$$

Now the final infinite product is the generating function for partitions into odd parts, and Theorem 1.5 follows from (4.11).

5. An Alternative Proof of Theorem 1.5.

Indeed, this entire study arose from examining the following two families of polynomials:

$$\sigma_n = \sum_{j\geq 0} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix},$$

and

$$\tau_n = \sum_{j\geq 0} q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix},$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0 & \text{if } B<0 \text{ or } B>A \\ \frac{(q)_A}{(q)_B(q)_{A-B}} & \text{if } 0\leq B\leq A. \end{cases}$$

Hence

$$\begin{aligned}\sigma_n - \sigma_{n-1} &= \sum_{j\geq 0} q^{\binom{j+1}{2}} \left(\begin{bmatrix} n-j \\ j \end{bmatrix} - \begin{bmatrix} n-j-1 \\ j \end{bmatrix} \right)\\ &= \sum_{j\geq 0} q^{\binom{j+1}{2}} q^{n-2j} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \text{ (by [And76, p. 35, eq. (3.3.3)])}\\ &= q^{n-1}\sum_{j\geq 0} q^{\binom{j}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix}\\ &= q^{n-1}\tau_{n-2},\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\tau_n - \sigma_n &= \sum_{j\geq 0} q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} \left(1-q^{1-j}\right) \\ &= \sum_{j\geq 0} q^{\binom{j}{2}} q^{n-2j} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} (1-q^{n-j}) \\ &= \sum_{j\geq 0} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} (1-q^{n-1-j}) \\ &= \sigma_{n-2} - q^{n-1}\tau_{n-2}.\end{aligned} \tag{5.13}$$

Adding (5.12) and (5.13), we find

$$\tau_n = \sigma_{n-1} + \sigma_{n-2}, \tag{5.14}$$

and substituting (5.14) into (5.12), we obtain

$$\sigma_n = \sigma_{n-1} + q^{n-1}(\sigma_{n-3} + \sigma_{n-4}). \tag{5.15}$$

Now (5.15) is precisely the recurrence that appears in (4.9). Additionally $\sigma_n = 0$ if $n < 0, \sigma_0 = \sigma_1 = 1, \sigma_2 = 1 + q$, and $\sigma_3 = 1 + q + q^2$. Hence σ_n fulfills the defining conditions for $S_3(n)$ given in (4.9). Therefore

$$\sigma_n = S_3(n).$$

Hence

$$\begin{aligned}\sum_{n\geq 0} \mathcal{L}_3(n) q^n &= \lim_{n\to\infty} S_3(n) = \lim_{n\to\infty} \sigma(n) \\ &= \lim_{n\to\infty} \sum_{j\geq 0} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} \\ &= \sum_{n\geq 0} \frac{q^{\binom{j+1}{2}}}{(q)_j} \\ &= (-q)_\infty \text{ (by [And76, p. 19, eq. (2.2.6)])} \\ &= \frac{1}{(q;q^2)_\infty} \text{ (by [And76, p. 5, eq. (1.2.5)])},\end{aligned}$$

and Theorem 1.5 follows as before.

6. Conclusion

One obvious project is suggested by these results: prove each of the Theorems bijectively.

A more subtle observation arises from the fact that by their very definition

$$\mathcal{L}_1(n) \geq \mathcal{L}_2(n) \geq \mathcal{L}_3(n).$$

Hence by Theorems 1.3-1.5, $B(n)$, the number of basis partitions of n, satisfies

$$p_4(n) \geq B(n) \geq p_2(n),$$

where $p_k(n)$ is the number of partitions of n in which no part is divisible by k. So, is there an inequality relating $B(n)$ and $p_3(n)$? In light of the fact that

$$\sum_{n\geq 0} (p_3(n) - B(n))q^n = q^4 + q^5 + q^6 + q^7 + 3q^8 + 3q^9 + 6q^{10} + \cdots + 1247q^{35} + \cdots,$$

we make the following:

Conjecture. $p_3(n) \geq B(n)$ with strict inequality if $n > 3$.

Finally, we note that since $\sigma_n = s_n$, consequently

$$\begin{aligned}\mathcal{S}_3(x) &= \sum_{n\geq 0} \frac{x^{3n}(1+x^2q^{2n+1})(-xq)_n q^{n(3n+1)/2}}{(x)_{n+1}} \\ &= \sum_{n\geq 0} \sigma_n x^n \\ &= \sum_{n\geq 0} x^n \sum_{n\geq 2j\geq 0} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} \\ &= \sum_{n,j\geq 0} x^{n+2n} q^{\binom{j+1}{2}} \begin{bmatrix} n+j \\ j \end{bmatrix} \\ &= \sum_{j\geq 0} \frac{x^{2j} q^{\binom{j+1}{2}}}{(x)_{j+1}} \text{ (by [And76, p. 36, eq. (3.3.7)])}\end{aligned} \tag{6.16}$$

a rather nice corollary from the two proofs of Theorem 1.5.

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