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Quantum q-series identities

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In memory of Srinivasa Ramanujan

Abstract. As analytic statements, classical q-series identities are equalities between power series for $|q| < 1$. This paper concerns a different kind of identity, which we call a quantum q-series identity. By a quantum q-series identity we mean an identity which does not hold as an equality between power series inside the unit disk in the classical sense, but does hold on a dense subset of the boundary – namely, at roots of unity. Prototypical examples were given over thirty years ago by Cohen and more recently by Bryson-Ono-Pitman-Rhoades and Folsom-Ki-Vu-Yang. We show how these and numerous other quantum q-series identities can all be easily deduced from one simple classical q-series transformation. We then use other results from the theory of q-hypergeometric series to find many more such identities. Some of these involve Ramanujan's false theta functions and/or mock theta functions.

Keywords. q-series identities, Ramanujan

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1. Introduction

Recall the usual q-series notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \tag{1.1}$$

valid for integers $n \geq 0$, and the limiting case

$$(a)_\infty = (a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}). \tag{1.2}$$

Let

$$\sigma(q) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(-q)_n} = 1 + q - q^2 + 2q^3 - 2q^4 + q^5 + q^7 - 2q^8 + \cdots \tag{1.3}$$

and

$$\sigma^*(q) = 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = -2q - 2q^2 - 2q^3 + 2q^7 + 2q^8 + 2q^{10} + \cdots . \tag{1.4}$$

The function $\sigma(q)$ is one of the most well-known q-series in Ramanujan's lost notebook. It appears there in the "sum-of-tails" identities [AnBe09, Entries 7.3.2, 7.3.3]

$$\frac{1}{2}\sigma(q) = \sum_{n\geq 0}((-q)_\infty - (-q)_n) + (-q)_\infty\left(\frac{1}{2} - \sum_{n\geq 1}\frac{q^n}{1-q^n}\right) \tag{1.5}$$

$$= \sum_{n\geq 0}\left((-q)_\infty - \frac{1}{(q;q^2)_{n+1}}\right) + (-q)_\infty\left(\frac{1}{2} - \sum_{n\geq 1}\frac{q^{2n}}{1-q^{2n}}\right), \tag{1.6}$$

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which have inspired a number of works – see [AF05, AJO01, Ch02, ChJi07], for example. However, Ramanujan's σ-function owes most of its fame to the unexpected fact, discovered by Andrews, Dyson, and Hickerson [ADH88], that its coefficients can be described using the arithmetic of the real quadratic field $\mathbb{Q}(\sqrt{6})$. Specifically, let $T(n)$ denote the number of inequivalent integer solutions to the equation

$$u^2 - 6v^2 = n \tag{1.7}$$

having $u + 3v \equiv \pm 1 \pmod{12}$ minus the number of such solutions having $u + 3v \equiv \pm 5 \pmod{12}$. Then Andrews, Dyson, and Hickerson showed that

$$\sigma(q) = \sum_{n \geq 0} T(24n+1) q^n. \tag{1.8}$$

They also showed that the function $\sigma^*(q)$ is a companion to $\sigma(q)$ in the sense that

$$\sigma^*(q) = \sum_{n \geq 1} T(1-24n) q^n. \tag{1.9}$$

Subsequently Cohen [Co88] observed that $\sigma(q)$ and $\sigma^*(q)$ are also companions in a different sense. To state his result, note that using standard q-series identities or simple combinatorial arguments, we have [And86, Co88]

$$\sigma(q) = 1 + \sum_{n \geq 0} (-1)^n q^{n+1} (q)_n = 2 \sum_{n \geq 0} (-1)^n (q)_n \tag{1.10}$$

and

$$\sigma^*(q) = -2 \sum_{n \geq 0} q^{n+1} (q^2; q^2)_n. \tag{1.11}$$

Therefore both $\sigma(q)$ and $\sigma^*(q)$ are well-defined at roots of unity, since for a given root of unity q the terms $(q)_n$ and $(q^2; q^2)_n$ vanish for n sufficiently large. For example, we have

$$\begin{aligned} \sigma(i) &= 2 \sum_{n=0}^{3} (-1)^n (i; i)_n \\ &= 2(1 - (1-i) + (1-i)(1-i^2) - (1-i)(1-i^2)(1-i^3)) \\ &= -4 - 2i \end{aligned}$$

while

$$\begin{aligned} \sigma^*(-i) &= -2 \sum_{n=0}^{1} (-i)^{n+1} (-1; -1)_n \\ &= -2((-i) + (-i)^2(1-(-1))) \\ &= 4 + 2i, \end{aligned}$$

and so $\sigma(i) = -\sigma^*(-i)$. Cohen proved that this is true in general – namely, if q is a root of unity then

$$\sigma(q) = -\sigma^*(q^{-1}). \tag{1.12}$$

He also left two similar identities as exercises for the reader,

$$\frac{1}{2} \sigma^*(q^{-1}) = \sum_{n \geq 0} (-q)_n \tag{1.13}$$

for q a primitive even root of unity and

$$-\frac{1}{2}\sigma^*(q) = \sum_{n\geq 0}(q;q^2)_n \tag{1.14}$$

for q a primitive odd root of unity.

Note that identities (1.12) – (1.14) are certainly not valid inside the unit disk since none of the right-hand sides is a convergent power series there. This is the essence of what we call a *quantum q-series identity* – an identity between q-hypergeometric series that holds at roots of unity but not as an equality of power series for $|q| < 1$. Cohen's identities appear to have been the only ones of this nature until recent work of Bryson, Ono, Pitman, and Rhoades [BOPR12], who proved that if q is a root of unity then

$$F(q) = U(q^{-1}), \tag{1.15}$$

where

$$F(q) = \sum_{n\geq 0}(q)_n \tag{1.16}$$

is the Kontsevich-Zagier function [Za01] and

$$U(q) = \sum_{n\geq 0}(q)_n^2 q^{n+1}. \tag{1.17}$$

Folsom, Ki, Vu and Yang [FKVY17] generalized (1.15) by proving that if q is a primitive kth root of unity and $x \in \mathbb{C}$ then

$$F_k(x, q^{-1}) = x^k U_k(-x, q), \tag{1.18}$$

where

$$F_k(x,q) = \sum_{n=0}^{k-1} x^{n+1}(xq)_n \tag{1.19}$$

and

$$U_k(x,q) = \sum_{n=0}^{k-1}(-xq)_n(-x^{-1}q)_n q^{n+1}. \tag{1.20}$$

The existing proofs of (1.12), (1.15), and (1.18) use what has been called a "delicate recursion" [BFOR17, p. 340]. In this paper we show that all three can be deduced from a single classical q-series transformation [GaRa04, p.28, Ex. 1.15 (ii)],

$$\sum_{n=0}^{N}\frac{(q^{-N})_n(b)_n z^n}{(q)_n(c)_n} = \frac{(c/b)_N b^N}{(c)_N}\sum_{n=0}^{N}\frac{(q^{-N})_n(b)_n(q/z)_n(z/c)^n}{(q)_n(bq^{1-N}/c)_n(-1)^n q^{\binom{n}{2}}}. \tag{1.21}$$

In fact, many other quantum identities can be deduced from (1.21). We record these in two batches, using the notation

$$f(q) =_q g(q) \tag{1.22}$$

if $f(q) = g(q)$ whenever q is a suitable root of unity and

$$f(q) =_{q^{-1}} g(q) \tag{1.23}$$

if $f(q) = g(1/q)$ whenever q is a suitable root of unity.

Proposition 1.1. *We have the following quantum q-series identities.*

$$\sum_{n\geq 0}(q;q^2)_n q^n =_{q^{-1}} \sum_{n\geq 0}(q;q^2)_n^2 q^{2n+1}, \tag{1.24}$$

$$\sum_{n\geq 0}(q^2;q^2)_n q^{n+1} =_{q^{-1}} \sum_{n\geq 0}(q)_{2n} q^{2n+1}, \tag{1.25}$$

$$-\sum_{n\geq 0}(-q)_n =_{q^{-1}} \sum_{n\geq 0}(q^2;q^2)_n q^{n+1}, \tag{1.26}$$

$$\sum_{n\geq 0}(q;q^2)_n =_{q^{-1}} \sum_{n\geq 0}(q)_{2n} q^{2n+1}, \tag{1.27}$$

$$-\sum_{n\geq 0}(-q;q^2)_n q^n =_{q^{-1}} \sum_{n\geq 0}(q^2;q^4)_n q^{2n+1}, \tag{1.28}$$

$$\sum_{n\geq 0}(-q^2;q^2)_n (-q)^{n+1} =_{q^{-1}} \sum_{n\geq 0}(-q)_{2n} q^{2n+1}, \tag{1.29}$$

$$\sum_{n\geq 0}(q;q^2)_n q^{2n+1} =_q 1, \tag{1.30}$$

$$\sum_{n\geq 0}(q)_n q^{n+1} =_q 1. \tag{1.31}$$

Proposition 1.2. *We have the following quantum q-series identities.*

$$\sum_{n\geq 0}\frac{(q)_n}{(-q)_n} =_{q^{-1}} 2q\sum_{n\geq 0}\frac{(q)_n^2 q^n}{(-q)_{n+1}}, \tag{1.32}$$

$$\sum_{n\geq 0}\frac{(q)_n(-1)^n}{(-q)_n} =_{q^{-1}} 2q\sum_{n\geq 0}\frac{(q^2;q^2)_n q^n}{(-q)_{n+1}}, \tag{1.33}$$

$$\sum_{n\geq 0}\frac{(q^2;q^2)_n^2 q^{2n+2}}{(q;q^2)_{n+1}} =_{q^{-1}} \sum_{n\geq 0}\frac{(q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{1.34}$$

$$\sum_{n\geq 0}\frac{(q)_{2n} q^{2n}}{(-q^2;q^2)_n} =_{q^{-1}} 2\sum_{n\geq 0}\frac{(q^2;q^2)_n q^{n+2}}{(-q^2;q^2)_{n+1}}, \tag{1.35}$$

$$\sum_{n\geq 0}\frac{(q)_{2n} q^{2n+2}}{(-q^2;q^2)_{n+1}} =_{q^{-1}} \frac{1}{2}\sum_{n\geq 0}\frac{(q^2;q^2)_n q^n}{(-q^2;q^2)_n}, \tag{1.36}$$

$$\sum_{n\geq 0}\frac{(q)_n q^n}{(-q)_n} =_q \frac{2}{1+q}, \tag{1.37}$$

$$\sum_{n\geq 0}\frac{(q)_n q^n}{(-q)_{n+1}} =_q \frac{1}{2}q^{-1} \tag{1.38}$$

$$\sum_{n\geq 0}\frac{(q^2;q^2)_n q^{2n+1}}{(-q^3;q^2)_n} =_q 1. \tag{1.39}$$

Before continuing, we make a few remarks. First, in Propositions 1.1 and 1.2 and throughout the paper it should be understood that in some cases the set of roots of unity is suitably restricted so that the series naturally truncate and so that there are no poles in the corresponding rational functions. For example, (1.24) and (1.32) are valid for odd roots of unity, in the first case so that the sums truncate and in the second case to avoid poles in the denominators of the summands. Second, (1.26) is equivalent to (1.13), Cohen's first exercise left to the reader, while his second exercise (1.14) follows from (1.25) and (1.27). Next, the left-hand side of (1.24) and the right-hand side of (1.28) are false theta functions, since [AnBe05, Entry 9.5.2]

$$\sum_{n\geq 0}(q;q^2)_n q^n = \sum_{n\geq 0}(-1)^n q^{3n^2+2n}(1+q^{2n+1}). \tag{1.40}$$

Finally, (1.28) and (1.29) contain the third order mock theta functions [AnBe18, p. 5]

$$\nu(-q) = \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q^2)_{n+1}} = \sum_{n\geq 0}(-q;q^2)_n q^n \tag{1.41}$$

and

$$\psi(-q) = \sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(-q;q^2)_n} = \sum_{n\geq 0}(-q^2;q^2)_n(-q)^{n+1}. \tag{1.42}$$

The rest of this paper is organized as follows. In the next section we derive all of the quantum identities recorded thus far using the transformation (1.21). It turns out that this transformation is only one of many classical q-series identities that can be used to give quantum identities. We take a brief tour of these in Sections 3 and 4. The proofs are straightforward, and the motivated reader will easily find many more identities than what we present here. We close in Section 5 with some remarks on quantum q-series identities in knot theory.

2. Quantum identities from the transformation (1.21)

We begin with short proofs of the identities of Cohen, Bryson-Ono-Pitman-Rhoades, and Folsom-Ki-Vu-Yang.

Proofs of identities (1.12), (1.15), *and* (1.18). Setting $N = N-1$ and letting $c \to 0$ in (1.21), we have

$$\sum_{n=0}^{N-1}\frac{(q^{1-N})_n(b)_n z^n}{(q)_n} = b^{N-1}\sum_{n=0}^{N-1}\frac{(q^{1-N})_n(b)_n(q/z)_n(z/b)^n q^{-n^2-n}}{(q)_n}. \tag{2.43}$$

If q is a primitive Nth root of unity, then for $n \leq N-1$ we have

$$\frac{(q^{1-N})_n}{(q)_n} = 1, \tag{2.44}$$

in which case (2.43) becomes the polynomial identity

$$\sum_{n=0}^{N-1}(b)_n z^n = b^{N-1}\sum_{n=0}^{N-1}(b)_n(q/z)_n(z/b)^n q^{-n^2-n}. \tag{2.45}$$

If we let $b = q$ and $z = -1$ we have

$$\sum_{n=0}^{N-1}(q)_n(-1)^n = \sum_{n=0}^{N-1}(q^2;q^2)_n(-1)^n q^{-n^2-2n-1}. \tag{2.46}$$

Replacing q by $1/q$ on the right-hand side of (2.46) and using

$$(aq^{-1};q^{-1})_n = (q/a;q)_n(-a)^n q^{-\binom{n+1}{2}}, \tag{2.47}$$

we obtain that for q a primitive Nth root of unity,

$$\sum_{n=0}^{N-1}(q)_n(-1)^n =_{q^{-1}} \sum_{n=0}^{N-1}(q^2;q^2)_nq^{n+1}. \tag{2.48}$$

Using our notation (1.23), this may be written

$$\sum_{n\geq 0}(q)_n(-1)^n =_{q^{-1}} \sum_{n\geq 0}(q^2;q^2)_nq^{n+1}, \tag{2.49}$$

which is (1.12). Similarly, if we let $b = q$ and $z = 1$ in (2.45) and use (2.47) on the right-hand side we have

$$\sum_{n\geq 0}(q)_n =_{q^{-1}} \sum_{n\geq 0}(q)_n^2q^{n+1}. \tag{2.50}$$

This is (1.15). Finally, if we let $b = xq$ and $z = x$ in (2.45) we find

$$\sum_{n=0}^{N-1}x^{n+1}(xq)_n = x^N\sum_{n=0}^{N-1}(xq)_n(q/x)_nq^{-n^2-2n-1}. \tag{2.51}$$

Letting $q = 1/q$ on the right-hand side of (2.51) and applying (2.47) gives (1.18).

Now that we have seen how quantum q-series identities follow from classical identities, the rest of the proofs in this paper will be straightforward. We turn to Proposition 1.1.

Proof of Proposition 1.1. Identities (1.24) – (1.31) follow from making appropriate substitutions in (2.45). For example, if q is a primitive Nth root of unity with N odd, then so is q^2, and in this case (2.45) gives

$$\sum_{n=0}^{N-1}(b;q^2)_nz^n = b^{N-1}\sum_{n=0}^{N-1}(b;q^2)_n(q^2/z;q^2)_n(z/b)^nq^{-2n^2-2n}. \tag{2.52}$$

Setting $b = q$ we have

$$\sum_{n=0}^{N-1}(q;q^2)_nz^n = \sum_{n=0}^{N-1}(q;q^2)_n(q^2/z;q^2)_nz^nq^{-2n^2-3n-1}, \tag{2.53}$$

or more precisely

$$\sum_{n=0}^{\frac{N-1}{2}}(q;q^2)_nz^n = \sum_{n=0}^{\frac{N-1}{2}}(q;q^2)_n(q^2/z;q^2)_nz^nq^{-2n^2-3n-1}, \tag{2.54}$$

since $1 - q^N = 0$. In any case, the specializations $z = q, 1$, and q^2 are (1.24), (1.27), and (1.30), respectively. The other identities are proved similarly.

We conclude this section with a proof of Proposition 1.2.

Proof of Proposition 1.2. We consider a different specialization of (1.21). Specifically, let $b = q$ and $N = N - 1$ to obtain

$$\sum_{n=0}^{N-1}\frac{(q^{1-N})_nz^n}{(c)_n} = \frac{(c/q)_{N-1}q^{N-1}}{(c)_{N-1}}\sum_{n=0}^{N-1}\frac{(q^{1-N})_n(q/z)_n(z/c)^n}{(q^{3-N}/c)_n(-1)^nq^{\binom{n}{2}}}. \tag{2.55}$$

Taking q to be an Nth root of unity and simplifying gives the rational function identity

$$\sum_{n=0}^{N-1}\frac{(q)_nz^n}{(c)_n} = \frac{(1-c/q)q^{-1}}{(1-c/q^2)}\sum_{n=0}^{N-1}\frac{(q)_n(q/z)_n(z/c)^n}{(q^3/c)_n(-1)^nq^{\binom{n}{2}}}. \tag{2.56}$$

The substitutions required to obtain (1.32) – (1.39) should be clear.

3. Further quantum identities, I

In the rest of the paper we apply the method of the previous section to several classical q-series identities in order to deduce quantum identities. In this section we use q-series summations and in Section 4 we use transformations.

3.A. Quantum identities from the q-Pfaff-Saalschutz summation

We begin with a proposition.

Proposition 3.1. *If q is an Nth root of unity, we have the rational function identity*

$$\sum_{n=0}^{N-1} \frac{(q)_n(b)_nq^n}{(c)_n(bq^3/c)_n} = \frac{(1-c/q)(1-c/bq^2)}{(1-c/q^2)(1-c/bq)}. \tag{3.57}$$

Proof. We recall the q-Pfaff-Saalschutz sum [GaRa04, p.355, Eq. (II.12)],

$$\sum_{n=0}^{N} \frac{(q^{-N})_n(a)_n(b)_nq^n}{(q)_n(c)_n(abq^{1-N}/c)_n} = \frac{(c/a)_N(c/b)_N}{(c)_N(c/ab)_N}. \tag{3.58}$$

Setting $N = N-1$ and $a = q$ we obtain

$$\sum_{n=0}^{N-1} \frac{(q^{1-N})_n(b)_nq^n}{(c)_n(bq^{3-N}/c)_n} = \frac{(c/q)_{N-1}(c/b)_{N-1}}{(c)_{N-1}(c/bq)_{N-1}}. \tag{3.59}$$

Taking q to be an Nth root of unity and simplifying gives the result.

The following quantum identities now follow easily.

Corollary 3.2. *We have*

$$\sum_{n\geq 0} \frac{(q)_n^2q^{n+1}}{(-q)_{n+1}^2} =_q \frac{1}{4}, \tag{3.60}$$

$$\sum_{n\geq 0} \frac{(q)_{2n}q^{2n+1}}{(-q^2)_{2n+1}} =_q \frac{1}{2}, \tag{3.61}$$

$$\sum_{n\geq 0} \frac{(q)_n^2q^{n+1}}{(-q^2;q^2)_{n+1}} =_q \frac{1}{2}. \tag{3.62}$$

3.B. Quantum identities from Jackson's summation

We begin with a proposition.

Proposition 3.3. *If q is an Nth root of unity, we have the rational function identity*

$$\sum_{n=0}^{N-1} \frac{(1-aq^{2n})(c)_n(q)_n(a/cq)^n}{(aq/c)_n(a)_n} = \frac{(1-a/q)(1-a/c)}{(1-a/cq)}. \tag{3.63}$$

Proof. We recall a summation of Jackson [GaRa04, p.356, Eq. (II.21)],

$$\sum_{n=0}^{N} \frac{(a)_n(1-aq^{2n})(b)_n(c)_n(q^{-N})_n}{(q)_n(1-a)(aq/b)_n(aq/c)_n(aq^{N+1})_n}\left(\frac{aq^{N+1}}{bc}\right)^n = \frac{(aq)_N(aq/bc)_N}{(aq/b)_N(aq/c)_N}. \tag{3.64}$$

Setting $N=N-1$ and $b=q$ we obtain

$$\sum_{n=0}^{N-1} \frac{(1-aq^{2n})(c)_n(q^{1-N})_n}{(aq/c)_n(aq^N)_n}\left(\frac{aq^{N-1}}{c}\right)^n = \frac{(1-aq^{N-1})(a/c)_{N-1}}{(aq/c)_{N-1}}. \tag{3.65}$$

Taking q to be an Nth root of unity and simplifying gives the result.

A few easy corollaries are listed below.

Corollary 3.4. *We have*

$$\sum_{n\geq 0} \frac{(q)_n(1+q^{2n+1})q^{\binom{n}{2}}}{(-q)_n} =_q 2, \tag{3.66}$$

$$\sum_{n\geq 0} \frac{(q)_n^2(1+q^{2n+1})(-q)^{-n}}{(-q)_n^2} =_q \frac{4}{1+q^{-1}}, \tag{3.67}$$

$$\sum_{n\geq 0} \frac{(q)_{2n}(1+q^{4n+2})(-q)^{-n-1}}{(-q^2)_{2n}} =_q -2. \tag{3.68}$$

4. Further quantum identities, II

In this section we deduce quantum q-series identities from transformations of Sears, Jain, Singh, and Watson.

4.A. Quantum identities from Sears' transformation

We begin with a proposition.

Proposition 4.1. *If q is a primitive Nth root of unity, we have the rational function identity*

$$\sum_{n=0}^{N-1} \frac{(b)_n(c)_nq^n}{(d)_n} = c^{N-1}\sum_{n=0}^{N-1} \frac{(c)_n(d/b)_n(b/c)^n}{(d)_n(-1)^nq^{\binom{n+1}{2}}}. \tag{4.69}$$

Proof. We recall a transformation of Sears [GaRa04, p.360, Eq. (III.12)],

$$\sum_{n=0}^{N} \frac{(q^{-N})_n(b)_n(c)_nq^n}{(q)_n(d)_n(e)_n} = \frac{c^N(e/c)_N}{(e)_N}\sum_{n=0}^{N} \frac{(q^{-N})_n(c)_n(d/b)_n}{(q)_n(d)_n(cq^{1-N}/e)_n}\left(\frac{bq}{e}\right)^n. \tag{4.70}$$

Setting $N=N-1$ and letting $e\to 0$ we obtain

$$\sum_{n=0}^{N-1} \frac{(q^{1-N})_n(b)_n(c)_nq^n}{(q)_n(d)_n} = c^{N-1}\sum_{n=0}^{N-1} \frac{(q^{1-N})_n(c)_n(d/b)_n(b/c)^n}{(q)_n(d)_n(-1)^nq^{\binom{n+1}{2}-Nn}}. \tag{4.71}$$

Taking q to be a primitive Nth root of unity and simplifying gives the result.

Corollary 4.2.

$$\sum_{n\geq 0} \frac{(q^2;q^4)_nq^{2n}}{(-q^2;q^2)_n} =_{q^{-1}} \sum_{n\geq 0} \frac{(q;q^2)_n^2q^{2n+1}}{(-q^2;q^2)_n}, \tag{4.72}$$

$$\sum_{n\geq 0} \frac{(-q)_{2n}q^{2n}}{(q;q^2)_n} =_{q^{-1}} -\sum_{n\geq 0} \frac{(-1;q^2)_n(-q^2;q^2)_nq^{2n+2}}{(q;q^2)_n}, \tag{4.73}$$

$$\sum_{n\geq 0} \frac{(q;q^2)_n^2q^{2n}}{(-q;q^2)_n} =_{q^{-1}} \sum_{n\geq 0} \frac{(q;q^2)_n(-1;q^2)_nq^{2n+1}}{(-q;q^2)_n}. \tag{4.74}$$

4.B. Quantum identities from a transformation of Jain

We begin with a two-part proposition.

Proposition 4.3. *If q is a primitive Nth root of unity, then we have the rational function identities*

$$\sum_{n=0}^{N-1} \frac{(q)_{2n}(q;q^2)_n q^{2n}}{(bq;q^2)_n (d)_{2n}} = \frac{q^{-1}(1-d/q)}{(1-d/q^2)} \sum_{n=0}^{N-1} \frac{(q)_n (b;q^2)_n (q/d)^n}{(b)_n (q^3/d)_n q^{\binom{n}{2}}} \tag{4.75}$$

and

$$\sum_{n=0}^{N-1} \frac{(q;q^2)_n (a)_{2n} q^{2n}}{(bq;q^2)_n} = a^{N-1} \sum_{n=0}^{N-1} \frac{(a)_n (b;q^2)_n (-a)^{-n} q^{-n^2}}{(b)_n}. \tag{4.76}$$

Proof. We recall a transformation of Jain [Ja81] (cf. [GaRa04, p. 101]),

$$\sum_{n=0}^{N} \frac{(q^{-N};q^2)_n (q^{1-N};q^2)_n (a)_{2n} q^{2n}}{(q^2;q^2)_n (bq;q^2)_n (d)_{2n}} = \frac{a^N (d/a)_N}{(d)_N} \sum_{n=0}^{N} \frac{(q^{-N})_n (a)_n (b;q^2)_n (q/d)^n}{(q)_n (b)_n (aq^{1-N}/d)_n q^{\binom{n}{2}}}. \tag{4.77}$$

Setting $N = N-1$ and $a = q$ we have

$$\sum_{n=0}^{N-1} \frac{(q^{1-N})_{2n} (q;q^2)_n q^{2n}}{(bq;q^2)_n (d)_{2n}} = \frac{q^{N-1}(1-d/q)}{(1-dq^{N-2})} \sum_{n=0}^{N-1} \frac{(q^{1-N})_n (b;q^2)_n (q/d)^n}{(b)_n (q^{3-N}/d)_n q^{\binom{n}{2}}}. \tag{4.78}$$

Letting q be an Nth root of unity and simplifying gives (4.75). Alternatively, if we set $N = N-1$ and let $d \to 0$ in (4.77), and then take q to be an Nth root of unity we have (4.76).

We collect some corollaries of Proposition 4.3 in two batches, corresponding to (4.75) and (4.76), respectively. Note that (4.84) contains Ramanujan's fifth order mock theta function [AnBe18, p. 18]

$$\psi_1(q) = \sum_{n \geq 0} (-q)_n q^{\binom{n+1}{2}}. \tag{4.79}$$

Corollary 4.4.

$$\sum_{n \geq 0} (q)_{2n} (q;q^2)_n q^{2n} =_{q^{-1}} q \sum_{n \geq 0} (q)_n q^{\binom{n+1}{2}}, \tag{4.80}$$

$$\sum_{n \geq 0} \frac{(q)_{2n} (q;q^2)_n q^{2n}}{(-q)_{2n}} =_{q^{-1}} 2q \sum_{n \geq 0} \frac{(q)_n q^{\binom{n+1}{2}}}{(-q)_{n+1}}, \tag{4.81}$$

$$\sum_{n \geq 0} \frac{(q)_{2n} (q;q^2)_n q^{2n}}{(-q^2;q^2)_n} =_{q^{-1}} \sum_{n \geq 0} \frac{(q)_n (-q;q^2)_n q^{n+1}}{(-q)_n}. \tag{4.82}$$

Corollary 4.5.

$$\sum_{n \geq 0} (q;q^2)_n (-q)_{2n} q^{2n} =_{q^{-1}} -q \sum_{n \geq 0} (-q)_n q^{\binom{n+1}{2}}, \tag{4.83}$$

$$\sum_{n \geq 0} (q;q^2)_n (-1)_{2n} q^{2n} =_{q^{-1}} -\sum_{n \geq 0} (-1)_n q^{\binom{n+1}{2}}, \tag{4.84}$$

$$\sum_{n \geq 0} (q^2;q^4)_n (-q;q^2)_{2n} q^{4n} =_{q^{-1}} -q \sum_{n \geq 0} (-q;q^2)_n q^{n^2+n}. \tag{4.85}$$

4.C. Quantum identities from a transformation of Singh

We begin again with a two-part proposition.

Proposition 4.6. *If q is a primitive Nth root of unity, then*

$$\sum_{n=0}^{N-1} \frac{(q)_n(a)_n(aq/b)_n q^n}{(a^2q^2/b;q^2)_n(-q^2)_n} = \sum_{n=0}^{N-1} \frac{(q^2;q^2)_n(a;q^2)_n(aq/b;q^2)_n q^{2n}}{(a^2q^2/b;q^2)_n(-q^2)_{2n}} \tag{4.86}$$

and

$$\sum_{n=0}^{N-1} \frac{(a)_n(aq/b)_n q^n}{(a^2q^2/b;q^2)_n} = \sum_{n=0}^{N-1} \frac{(a;q^2)_n(aq/b;q^2)_n q^{2n}}{(a^2q^2/b;q^2)_n}. \tag{4.87}$$

Proof. We recall a transformation of Singh [Si59] (cf. [GaRa04, Eq. (3.10.12)])

$$\sum_{n=0}^{N} \frac{(q^{-N})_n(a)_n(aq/b)_n(-aq/w)_n q^n}{(q)_n(a^2q^2/b;q^2)_n(aq^{1-N}/w)_n} = \sum_{n=0}^{N} \frac{(q^{-2N};q^2)_n(a;q^2)_n(aq/b;q^2)_n(a^2q^2/w^2;q^2)_n q^{2n}}{(q^2;q^2)_n(a^2q^2/b;q^2)_n(aq^{1-N}/w;q^2)_n(aq^{2-N}/w;q^2)_n}. \tag{4.88}$$

Setting $N = N-1$ and $w = -a$ we obtain

$$\sum_{n=0}^{N-1} \frac{(q^{1-N})_n(a)_n(aq/b)_n q^n}{(a^2q^2/b;q^2)_n(-q^{2-N})_n} = \sum_{n=0}^{N-1} \frac{(q^{2-2N};q^2)_n(a;q^2)_n(aq/b;q^2)_n q^{2n}}{(a^2q^2/b;q^2)_n(-q^{2-N})_{2n}}. \tag{4.89}$$

Letting q be an Nth root of unity and simplifying gives (4.86). Alternatively, if we set $N = N-1$ and let $w \to \infty$ in (4.88), and then take q to be an Nth root of unity we have (4.87).

We have the following corollaries.

Corollary 4.7.

$$\sum_{n\geq 0} \frac{(q^2;q^2)_n q^{2n}}{(-q)_{2n+1}} =_q \frac{1}{2}q^{-1}, \tag{4.90}$$

$$\sum_{n\geq 0} \frac{(q)_n^2 q^n}{(-q)_{n+1}} =_q \sum_{n\geq 0} \frac{(q)_{2n} q^{2n}}{(-q)_{2n+1}}, \tag{4.91}$$

$$\sum_{n\geq 0} \frac{(q)_n(q^2;q^2)_n q^n}{(-q;q^2)_{n+1}(-q)_{n+1}} =_q \sum_{n\geq 0} \frac{(q)_{2n} q^{2n}}{(-q)_{2n+1}(1+q^{2n+1})}. \tag{4.92}$$

Corollary 4.8.

$$\sum_{n\geq 0} \frac{(-q)_n^2 q^n}{(q;q^2)_{n+1}} =_q \sum_{n\geq 0} \frac{(-q;q^2)_n^2 q^{2n}}{(q;q^2)_{n+1}}, \tag{4.93}$$

$$\sum_{n\geq 0} \frac{(q;q^2)_n q^n}{(-q^2;q^2)_n} =_q \sum_{n\geq 0} \frac{(q;q^4)_n q^{2n}}{(-q^2;q^2)_n}, \tag{4.94}$$

$$\sum_{n\geq 0} \frac{(-1)_n^2 q^n}{(q;q^2)_n} =_q \sum_{n\geq 0} \frac{(-1;q^2)_n^2 q^{2n}}{(q;q^2)_n}. \tag{4.95}$$

We note that (4.90) uses the case $a = 0$ of (4.86) together with (1.38). Also, the left-hand side of (4.94) is a false theta function [AnBe09, Entry (9.3.1), $(a,q) = (1,-q)$],

$$\sum_{n\geq 0} \frac{(q;q^2)_n q^n}{(-q^2;q^2)_n} = \sum_{n\geq 0} (-1)^{\binom{n}{2}} q^{\binom{n+1}{2}}. \tag{4.96}$$

4.D. Quantum identities from a transformation of Watson

Our final proposition is the following.

Proposition 4.9. *If q is an Nth root of unity, then*

$$\begin{aligned}\sum_{n=0}^{N-1}&\frac{(1-aq^{2n})(b)_n(c)_n(e)_n(q)_n}{(aq/b)_n(aq/c)_n(aq/e)_n(a)_n}\left(\frac{a^2}{bce}\right)^n\\&=\frac{(1-a/q)(1-a/e)}{(1-a/qe)}\sum_{n=0}^{N-1}\frac{(aq/bc)_n(e)_n(q)_nq^n}{(aq/b)_n(aq/c)_n(eq^2/a)_n}.\end{aligned}\tag{4.97}$$

Proof. We recall Watson's transformation [GaRa04, p. 360, Eq. (III.18)],

$$\begin{aligned}\sum_{n=0}^{N}&\frac{(a)_n(1-aq^{2n})(b)_n(c)_n(d)_n(e)_n(q^{-N})_n}{(q)_n(1-a)(aq/b)_n(aq/c)_n(aq/d)_n(aq/e)_n(aq^{N+1})_n}\left(\frac{a^2q^{N+2}}{bcde}\right)^n\\&=\frac{(aq)_N(aq/de)_N}{(aq/d)_N(aq/e)_N}\sum_{n=0}^{N}\frac{(aq/bc)_n(d)_n(e)_n(q^{-N})_nq^n}{(q)_n(aq/b)_n(aq/c)_n(deq^{-N}/a)_n}.\end{aligned}\tag{4.98}$$

Setting $N=N-1$ and $d=q$ we obtain

$$\begin{aligned}\sum_{n=0}^{N-1}&\frac{(1-aq^{2n})(b)_n(c)_n(e)_n(q^{1-N})_n}{(aq/b)_n(aq/c)_n(aq/e)_n(aq^N)_n}\left(\frac{a^2q^N}{bce}\right)^n\\&=\frac{(1-aq^{N-1})(1-a/e)}{(1-aq^{N-1}/e)}\sum_{n=0}^{N}\frac{(aq/bc)_n(e)_n(q^{1-N})_nq^n}{(aq/b)_n(aq/c)_n(eq^{2-N}/a)_n}.\end{aligned}\tag{4.99}$$

Letting q be an Nth root of unity gives the result.

Corollary 4.10.

$$2\sum_{n\geq 0}(q)_n(-1)^n =_q \sum_{n\geq 0}\frac{(q)_n(1+q^{2n+1})(-1)^nq^{n(3n+1)/2}}{(-q)_n},\tag{4.100}$$

$$2\sum_{n\geq 0}\frac{(q)_n(-1)^n}{(-q)_n} =_q \sum_{n\geq 0}\frac{(q)_n^2(1+q^{2n+1})q^{n^2}}{(-q)_n^2},\tag{4.101}$$

$$\sum_{n\geq 0}\frac{(q)_n(-1)^nq^{\binom{n+1}{2}}}{(-q)_n} =_q 2\sum_{n\geq 0}\frac{(q^2;q^2)_n(-1)^n}{(-q;q^2)_{n+1}}\tag{4.102}$$

Note that the left-hand side of (4.100) is Ramanujan's $\sigma(q)$, while the left-hand side of (4.102) is a q-series with similar properties to $\sigma(q)$ studied by Corson, Favero, Liesinger, and Zubairy [CFLZ04].

5. Concluding Remarks

Though it has not been our focus here, we wish to point out that quantum q-series identities have important connections to quantum modular forms [BOPR12, Za10] and quantum topology [HiLo15, LoOs21, LoOs19]. In the latter case, colored Jones polynomials can be used to prove quantum identitites more involved than the ones in this paper. For example, corresponding to the family of torus knots $T(2,2t+1)$ we have the following generalization of (1.15) [HiLo15],

$$\sum_{k_t \geq \cdots \geq k_1 \geq 1} (q;q)_{k_t-1}^2 q^{k_t} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1} k_j \\ k_{i+1}-k_i \end{bmatrix}$$
$$=_{q^{-1}} \sum_{k_t \geq \cdots \geq k_1 \geq 0}^{\infty} (q;q)_{k_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}. \tag{5.103}$$

Here we have used the usual q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

A similar generalization of (1.15) arises from the family of twist knots K_p [LoOs21, Corollary 1.3, $m=1$],

$$\sum_{n_{2p-1} \geq \cdots \geq n_1 \geq 0} (q)_{n_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{n_i} q^{\binom{n_i}{2}-n_i n_{i+1}} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$
$$=_{q^{-1}} \sum_{n_p \geq \cdots \geq n_1 \geq 0} (q)_{n_p}^2 q^{n_p+1} \prod_{j=1}^{p-1} q^{n_j^2+n_j} \begin{bmatrix} n_{j+1} \\ n_j \end{bmatrix}. \tag{5.104}$$

It would be interesting to see q-series proofs of (5.103) and (5.104).

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