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ON A THEOREM OF ERDÖS AND SZEMEREDI

By M. J. NARLIKAR

§ 1. Introduction

In 1951, K. F. Roth [7] proved that if \( 1 = q_1 < q_2 < \ldots \) is the sequence of all square-free integers, then

\[
q_n + 1 - q_n = O \left( n^{\frac{3}{13}} (\log n)^{\frac{4}{13}} \right)
\]

and this was improved to \( O \left( n^{\frac{2}{9}} \right) \) by H. E. Richert [6]. In an attempt to put these in more general setting P. Erdős [1] introduced with any sequence \( B : 2 < b_1 < b_2 < \ldots \) the sequence \( Q : 1 - q_1 < q_2 \ldots \) of all integers \( q_i \) not divisible by any \( b_j \) and proved (subject only to \( (b_i, b_j) = 1 \) unless \( i = j \) and \( \sum \frac{1}{b_i} < \infty \) that

\[
q_n + 1 - q_n = O \left( q_n^{\theta} \right),
\]

with some \( \theta < 1 \), where \( \theta \) is independent of \( B \). His \( \theta \) was close to 1. E. Szemeredi made an important progress and showed that this is true for every fixed \( \theta > \frac{1}{2} \). As in all previous results of this kind, he showed that if \( Q(x) = \sum q_i \leq x \),

then \( Q(x+h) - Q(x) \gg h \), where \( h > \zeta^{\theta} \).
Using the ideas of Szemerédi with some refinements, we prove

**Theorem 1**

Let \( p \) be any prime and \( r_p \) denote the number of \( b_i \) divisible by \( p \) and suppose that as \( p \) varies, \( r_p \) does not exceed \( p^A \) where \( A \) is any positive constant. Let \( \sum \frac{1}{b_i} < \infty \). Then

\[
Q(x + h) - Q(x) \gg h, \text{ where } x > h > x^\theta
\]

and \( \theta > \frac{1}{2} \) is any constant.

Further if for some \( \lambda < 1 \), we have \( \sum b_i^{-\lambda} < \infty \), then

\[
Q(x + h) - Q(x) \gg h
\]

where \( h > x^\theta \), and \( \theta > \frac{\alpha}{1 + \alpha} = \beta \) say.

Next using the ideas of Jutila [4], the results of Brun [1] and the zero-free region for \( \zeta(s) \) due to I M. Vinogradov [9] we prove

**Theorem 2**

Let \( r_p \leq p^4 \) as before and in place of

\[
\sum b_i^{-1} = \infty \text{ let } \lim_{y \to \infty} \frac{\sum b_i^{-1} = \theta}. \text{ Then}
\]

\[
Q(x + h) - Q(x) \gg h / \nu_3 x,
\]

where \( h > x^\theta \), with \( \theta > \frac{1}{4} \).

The improvement Theorem 2 of Theorem 1 was suggested by Professor K. Ramachandra and I am thankful to him for explaining the same. Also I express my gratitude to him for encouragement and useful suggestions. He and I, in a joint paper to appear have improved Theorem 2 in several ways. These researches will appear in Acta Arithmetica in due course.
§ 2. Proofs of Theorems 1 & 2

We begin with some notations.

1. We can assume without loss of generality that $b_1, b_2, \ldots, b_k$ are primes (because we can replace them by their greatest prime factor and select distinct ones amongst them). Next we assume

$$\sum_{i = k}^{j_0} b_i^{-1} < \frac{1}{2}$$

and define $j_0$ by

$$\sum_{i > j_0} b_i^{-1} < \eta,$$ where $\eta$ is sufficiently small.

2. Let $n > 10$ be any large integer constant and for $i = 1, 2, \ldots, n$ put

$$C_i = \{ p | x (2n) - 1 + (i - 1) (8n^4) - 1 < p \leq x (2n) - 1 + i (8n^4) - 1 \},$$

$$C_i' = \{ p | x (2n) - 1 + (i - 1) (8n^4) - 1 < p \leq x (2n) - 1 + i (8n^4) - 1 \}.$$

Let $g$ run over integers of the form $\pi p$, where $p$ are chosen one from each $C_i$. Let $g_i'$ be defined in the same way with respect to $C_i'$. For any fixed integer $g$ the number of integers in $[x, x + h]$ which are divisible by $g$ (respectively $g b_i$), $k < i < j_0$ but coprime to $b_1, b_2, \ldots, b_k$ is

$$\frac{h}{g} \pi \left( \frac{1}{b_i} \right) + O \left( 2^k \right),$$

(respectively $\frac{h}{g b_j} \pi \left( \frac{1}{b_i} \right) + O \left( 2^k \right)$)
Also the number of integers in $[x, x+h]$ divisible by $b_i$ for any fixed $i > j_0$ such that $b_i < h$ is $\leq \frac{2h}{b_i}$. Hence the number of integers (counted with certain multiplicities) in $[x, x+h]$ which are coprime to $b_1, b_2, \ldots, b_k$, but divisible by some $g$ or other, but not divisible by any $b_i (i > j_0, b_i < h)$

\[
> \sum_{g} \left\{ \frac{h}{g} \sum_{i < k} \left( 1 - \frac{1}{b_i} \right) + O \left( \frac{1}{b_i} \right) \right\} - \sum_{b_{j_0} < b_j < h} \frac{2h}{b_i} \\
> \sum_{g} \left( \frac{h}{2g} \sum_{i < k} \left( 1 - \frac{1}{b_i} \right) \right) - \sum_{b_{j_0} < b_i} 2h \\
= h \left\{ \frac{1}{2} \left( \prod_{p \in C_i} \frac{1}{p} \right) \sum_{i < k} \left( 1 - \frac{1}{b_i} \right) - 2\eta \right\} + O \left( 2^k \frac{1}{x^2} + \frac{1}{8n^2} \right) \\
> h \left\{ \left( \frac{1}{1000n} \right)^{10n} \sum_{i < k} \left( 1 - \frac{1}{b_i} \right) - 2\eta \right\} + O \left( 2^k \frac{1}{x^2} + \frac{1}{8n^2} \right)
\]
The multiplicities are \( \leq (4n) 1 \) since the number of prime factors \( \geq x^{2n} \) is \( \leq 4n \) for the integers counted. Now let us look at integers in \([x, x + h]\) which are of the form \( mg \) but divisible by some \( b_i \) or the other with \( b_i > h \). Now a given \( b_i \) can divide at most one integer in the interval and so it suffices to count the number of \( b_i \) to get an upper bound for the number of integers in question. If \( (b_i, g) = 1 \), then \( b_i g \leq 2x \) and hence \( hg \leq 2x \). This is impossible since \( h > x^\theta \). Hence \( (b_i, g) > 1 \) and the number of possible \( b_i \)'s is therefore

\[
\leq \sum_{p \leq x^n} \frac{1}{p^A} < x^n
\]

A large choice of \( n \) now completes the proof of the first part of Theorem 1. The second part can be proved similarly using \( C_i' \).

The proof of Theorem 2 to put it briefly starts with

\[
\frac{1}{2\pi i} \int \left( -\sum_p (p^{-s} \log p) \right) \left( \sum_{X < p \leq 2X} p^{-s} \right)^N x \frac{(x + h)^s - x^s}{s} ds
\]

where the line of integration is \( \sigma = 1 + (\log x)^{-1} \), \( |t| < T \). We then move the line of integration to \( \sigma = 1 - (\log T)^{-100} \). Rough estimations are enough to
show that the number of numbers of the form \( p_1 p_2 \cdots p_N \)

\( (X < p_i < 2X) \) lying in \([x, x + h]\) is \( \gg \frac{h}{\log x} \) (provided

\[ \frac{1}{n} \text{ and } N = n - 1 \text{ and } h = x^2 \text{ where } \eta > 0 \text{ is small provided } n \text{ is large}. \]

If \( b_i \) divides a number of the counted type then \( x^{2n} < b_i < x^2 \) and by Brun's sieve the number of counted numbers divisible by such \( b_i \) is,

\[
\ll \sum \left( \frac{1}{x^{2n}} \leq b_i \leq \frac{h}{b_i \log x} + 1 \right) + \sum \left( \frac{h}{b_i \log x} \leq b_i \leq \frac{1}{(\log x)^2} \right) + \sum \left( \frac{h}{(\log x)^2} \leq b_i \leq x^2 \right),
\]

where the second sum is over those \( b_i \) which divide a number of the counted type. This proves Theorem 2.

Remark: In the joint paper [5] to appear, the present paper has been referred to under the title "An Analytic Approach to Szemerédi’s Theorem". The interested readers will please note this change of title. The paper was written in 1977 and could not be published earlier because of some reasons.
References


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