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ON A THEOREM OF ERDÖS AND SZEMEREDI

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§ 1. Introduction

In 1951, K. F. Roth [7] proved that if $1 = q_1 < q_2 < \ldots$ is the sequence of all square - free integers, then

$$q_{n+1} - q_n = O\left(n^{\frac{3}{13}} (\log n)^{\frac{4}{13}}\right)$$

and this was improved to $O\left(n^{\frac{2}{9}}\right)$ by H. E Richert [6]. In an attempt to put these in more general setting P. Erdös [1] introduced with any sequence B : $2 \leqslant b_1 < b_2 < \ldots\ldots$, the sequence $Q : 1 = q_1 < q_2 \cdot$ of all integers q_i not divisible by any b_j and proved (subject only to $(b_i, b_j) = 1$ unless $i = j$ and $\sum \frac{1}{b_i} < \infty$) that

$$q_{n+1} - q_n = O\left(q_n^{\theta}\right),$$

with some $\theta < 1$, where θ is independent of B. His θ was close to 1. E. Szemeredi made an important progress and showed that this is true for every fixed $\theta > \frac{1}{2}$. As in all previous results of this kind, he showed that if $Q(x) = \sum_{q_i \leqslant x} 1$, then $Q(x + h) - Q(x) \gg h$, where $h > x^{\theta}$.

Using the ideas of Szemeredi with some refinements, we prove

Theorem 1

Let p be any prime and r_p denote the number of b_i divisible by p and suppose that as p varies, r_p does not exceed p^A where A is any positive constant. Let $\sum \frac{1}{b_i} < \infty$. Then

$$Q(x+h) - Q(x) \gg h, \text{ where } x \geqslant h \geqslant x^{\theta}$$

and $\theta > \frac{1}{2}$ is any constant.

Further if for some $\alpha < 1$, we have $\sum b_i^{-\alpha} < \infty$, then

$$Q(x+h) - Q(x) \gg h.$$

where $h > x^{\theta}$, and $\theta > \frac{\alpha}{1+\alpha} = \beta$ say.

Next using the ideas of Jutila [4], the results of Brun [1] and the zero-free region for $\zeta(s)$ due to I M. Vinogradov [9] we prove

Theorem 2

Let $r_p \leqslant p^A$ as before and in place of

$$\sum b_i^{-1} < \infty \text{ let } \lim_{y \to \infty} \sum_{y \leqslant b_i \leqslant y^2} b_i^{-1} = 0. \text{ Then}$$

$$Q(x+h) - Q(x) \gg h/\log x,$$

where $h > x^{\theta}$, with $\theta > \frac{1}{2}$.

The improvement Theorem 2 of Theorem 1 was suggested by Professor K. Ramachandra and I am thankful to him for explaining the same. Also I express my gratitude to him for encouragement and useful suggestions. He and I, in a joint paper to appear have improved Theorem 2 in several ways These researches will appear in Acta Arithmetica in due course.

§ 2. Proofs of Theorems 1 & 2

We begin with some notations.

(1) We can assume without loss of generality that $b_1, b_2, \ldots b_k$ are primes (because we can replace them by their greatest prime factor and select distinct ones amongst them). Next we assume

$$\sum_{i > k} b_i^{-1} < \tfrac{1}{2} \text{ and define } j_0 \text{ by}$$

$$\sum_{i > j_0} b_i^{-1} < \eta, \text{ where } \eta \text{ is sufficiently small.}$$

(2) Let $n > 10$ be any large integer constant and for $i = 1, 2, \ldots n$ put

$$C_i = \{ p \,/\, x^{(2n)^{-1} + (i-1)(8n^4)^{-1}} < p \leq x^{(2n)^{-1} + i(8n^4)^{-1}} \},$$

$$C_i' = \{ p \,/\, x^{\beta (2n)^{-1} + (i-1)(8n^4)^{-1}} < p \leq x^{\beta (2n)^{-1} + i(8n^4)^{-1}} \}.$$

Let g run over integers of the form $\prod_i p$, where p are chosen one from each C_i. Let g_i' be defined in the same way with respect to C_i'. For any fixed integer g the number of integers in $[x, x+h]$ which are divisible by g (respectively $g\, b_i$), $k < i \leq j_0$ but coprime to $b_1, b_2, \ldots b_k$ is

$$\frac{h}{g} \prod_{i \leq k} \left(1 - \frac{1}{b_i}\right) + O(2^k).$$

$$\left(\text{respectively } \frac{h}{g\, b_j} \prod_{i \leq k} \left(1 - \frac{1}{b_i}\right) + O(2^k) \right)$$

Also the number of integers in $[x, x+h]$ divisible by b_i for any fixed $i > j_0$ such that $b_i < h$ is $\leqslant \frac{2h}{b_i}$. Hence the number of integers (counted with certain multiplicities) in $[x, x+h]$ which are coprime to $b_1, b_2, \dots b_k$, but divisible by some g or other, but not divisible by any b_i $(i > j_0, b_i \leqslant h)$

$$> \sum_g \left\{ \frac{h}{g} \prod_{i \leqslant k} \left(1 - \frac{1}{b_i}\right) - \sum_{k < i \leqslant j_0} \left(\frac{h}{g\, b_i} \prod_{i \leqslant k} \left(1 - \frac{1}{b_i}\right) + O(2^k) \right) \right\}$$

$$- \sum_{b_{j_0} \leqslant b_j \leqslant h} \frac{2h}{b_i}$$

$$> \sum_g \left(\frac{h}{2g} \prod_{i \leqslant k} \left(1 - \frac{1}{b_i}\right) \right) - \sum_{b_{j_0} \leqslant b_i} \frac{2h}{b_i}$$

$$+ O\left(2^k j_0 \sum_g 1 \right)$$

$$= h \left\{ \frac{1}{2} \left(\prod_i \left(\sum_{p \in C_i} \frac{1}{p} \right) \right) \prod_{i \leqslant k} \left(1 - \frac{1}{b_i}\right) - 2\eta \right\}$$

$$+ O\left(2^k j_0 x^{\frac{1}{2} + \frac{1}{8n^2}}\right)$$

$$> h \left\{ \left(\frac{1}{1000n} \right)^{10n} \prod_{i \leqslant k} \left(1 - \frac{1}{b_i}\right) - 2\eta \right\}$$

$$+ O\left(2^k j_0 x^{\frac{1}{2} + \frac{1}{8n^2}}\right)$$

The multiplicities are $\leqslant (4n)!$ since the number of prime factors $> x^{\frac{1}{2n}}$ is $\leqslant 4n$ for the integers counted. Now let us look at integers in $[x, x+h]$ which are of the form $m\,g$ but divisible by some b_i or the other with $b_i > h$. Now a given b_i can divide atmost one integer in the interval and so it suffices to count the number of b_i to get an upper bound for the number of integers in question If $(b_i, g) = 1$, then $b_i\,g \leqslant 2x$ and hence $hg \leqslant 2x$. This is impossible since $h > x^{\theta}$. Hence $(b_i, g) > 1$ and the number of possible b_i's is therefore

$$\leqslant \sum_{p \leqslant x^{\frac{1}{n}}} p^A \leqslant x^{\frac{A+1}{n}}$$

A large choice of n now completes the proof of the first part of Theorem 1. The second part can be proved similarly using C_i'.

The proof of Theorem 2 to put it briefly starts with

$$\frac{1}{2\pi i}\int \Big(-\sum_{p}(p^{-s}\log p)\Big)\Big(\sum_{X \leqslant p \leqslant 2X} p^{-s}\Big)^{N} \times \frac{(x+h)^s - x^s}{s}\,ds$$

where the line of integration is $\sigma = 1 + (\log x)^{-1}$, $|t| \leqslant T$. We then move the line of integration to $\sigma = 1 - (\log T)^{-\frac{99}{100}}$. Rough estimations are enough to show that the number of numbers of the form $p\,p_1\,p_2 \cdots\cdots p_N$ $(X < p_i \leqslant 2X)$ lying in $[x, x+h]$ is $\gg \frac{h}{\log x}$ (provided $X = T^{\frac{1}{n}}$ and $N = n - 1$ and $h = x^{\frac{1}{2}+\eta}$ where $\eta > 0$ is small provided n is large). If b_i divides a number of the counted type then $x^{\frac{1}{2n}} \leqslant b_i < x^2$ and by Brun's sieve the number of counted numbers divisible by such b_i is

$$\ll \sum_{x^{\frac{1}{2n}} < b_i < \frac{h}{(\log x)^2}} \left(\frac{h}{b_i \log x} + 1 \right) + \sum_{\frac{h}{(\log x)^2} < b_i \leqslant X^2}' 1 ,$$

where the second sum is over those b_i which divide a number of the counted type. This proves Theorem 2.

Remark : In the joint paper [5] to appear, the present paper has been referred to under the title "**An Analytic Approach to Szemeredi's Theorem**". The interested readers will please note this change of title. The paper was written in 1977 and could not be published earlier because of some reasons.

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