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PROGRESS TOWARDS A CONJECTURE ON THE MEAN-VALUE OF TITCHMARSH SERIES-II

By K. RAMACHANDRA

§ 1. Introduction

The result of this paper may be considered as complementary to that of my earlier paper [2], on Titchmarsh series. Although not as interesting as the earlier result, the result of the present paper finds a nice application, (See [1]). In [3] I defined a class of series called Titchmarsh series and I now start by recalling its definition.

Titchmarsh Series. (or briefly K D T series).

Let \( A > 10 \) be a constant.

Let \( \frac{1}{A} < \mu_1 < \mu_2 < ... \) where

\[
\frac{1}{A} < \mu_{n+1} - \mu_n < A \quad (\text{for } n = 1, 2, 3, \ldots).
\]

(In [2] the notation is slightly different and we have used there \( \lambda_n \) instead of \( \mu_n \) and for simplicity assumed \( \lambda_1 = 1 \). Also we have used there \( a_1, a_2, \ldots \) in place of our present \( b_1, b_2, \ldots \) and assumed for simplicity \( a_1 = 1 \). We have written there \( F(s) \) instead of \( F_0(s) \).) Let \( b_1, b_2, \ldots \) be a sequence of complex numbers possibly depending on a parameter \( H > 10 \) such that

\[ |b_n| < (\mu_n H)^A. \]

Put \( F_0(s) = \sum_{n=1}^{\infty} b_n \frac{\mu_n - s}{n!} \) where \( s = \sigma + it \).
$F_0(s)$ is called a KDT series if there exists a constant $A > 10$ and a system of rectangles $R(T, T + H)$ defined by \{ $\sigma > 0$, $T < t < T + H$ \} where $10 < H < T$, and $T$ (which may be related to $H$) tends to infinity, and $F_0(s)$ admits an analytic continuation into these rectangles and the maximum of $|F_0(s)|$ taken over $R(T, T + H)$ does not exceed $\text{Exp}(HA)$.

I then made the following conjecture

**Conjecture**

For a KDT series $F_0(s)$, we have,

$$\frac{1}{H} \int_{L}^{H} |F(it)|^2 \, dt > C_A \sum_{n} b_n^2,$$

where $X = 2 + D_A H$, $L$ denotes the side ($\sigma = 0$, $T < t < T + H$) of $R(T, T + H)$, and $C_A$ and $D_A$ are positive constants depending only on $A$, provided $\mu_1 = b_1 = 1$.

I proved the following theorem.

**Theorem 1**

Under the restrictions $\mu_1 = b_1 = 1$, we have,

$$\frac{1}{H} \int_{L}^{H} |F_0(it)|^2 \, dt > C_A \sum_{n} b_n^2 < \left( 1 - \frac{\log \mu_{n}}{\log H} + \frac{1}{\log \log H} \right),$$

where $X = 2 + D_A H$ and, $C_A$ and $D_A$ are effective positive constants depending only on $A$.

I now prove the following theorem.
Theorem 1

For some convenience let us assume in the definition of Titchmarsh series $F_0(s)$ the rectangles $R(T, T + H)$ to be $(\sigma > \beta, T < t < T + H)$ where $\beta$ is a positive constant such that $0 < \beta < \alpha_1 < \frac{1}{2}$, where $\alpha_1$ is another constant. Let $k \geq 2$ be an integer and write $F(s) = (F_0(s))^k = \sum_{n=1}^{\infty} \frac{a_n \lambda_n^{-s}}{n^k}$, a series which is surely convergent where $F_0(s)$ is absolutely convergent. Put $Y = (M + H)$ where

$$\lambda = kA \quad (\alpha_1 - \beta)$$

and $M = \max$ of $|F_0(s)|$ taken over $R(T, T + H)$. Define the entire function $\phi(s)$ by

$$\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta \left( \frac{Y}{\lambda_n} \right)$$

where for $X > 0$, $\Delta(X)$ is defined by

$$\Delta(X) = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} X^W \exp \left( \frac{4a+2}{W} \right) \frac{dW}{W},$$

$a$ being a suitable positive integer constant at our choice. We now suppose that $\alpha, \beta, \alpha_1, \alpha_2, \alpha_3$ are constants satisfying

$$\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}.$$ 

Put $X = [2 - A - 100 H] + 2$ and

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} |a_n| 2 \left( \frac{H}{\mu_n} \right)^{2\sigma}.$$
Then we have,

\[ \frac{1}{H} \int_{L} |\phi(\alpha + it)|^{2k-1} dt \]

\[ \gg V(\alpha) H^{1-2\alpha} \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\} \]

provided only that \( V(\alpha) \neq 0 \) and \( (\log V(\alpha))(\log H)^{-1} \) is bounded below by a negative constant. Here the constant implied by the Vinogradov symbol \( \gg \) depends only on \( A, k, \alpha, \beta, \alpha_1, \alpha_2, \alpha_3 \) and the negative constant referred to just now. Further it is effective.

Remark: It will be clear from the proof that if

\[ \mu_n = n (n = 1, 2, 3, \ldots) \] then we can choose \( X = \left[ \frac{H}{100} \right] + 2. \)

The object of this paper is to prove theorem 2. The proof of the theorem is fairly long. The proof depends upon a special case of a convexity theorem of R. M. Gabriel which we state below (in the notation of D. R. Heath-Brown's paper [2]; for the more general theorem of Gabriel see the reference in [2] or Titchmarsh's famous book [5] pages 203 and 337.)

**Theorem 3**

Let \( f(z) \) be regular in the infinite strip \( \lambda < \text{Re} z < \beta \) and continuous for \( \lambda < \text{Re} z < \beta \). Suppose \( f(z) \to 0 \) as \( |\text{Im} z| \to \infty \), uniformly in \( \lambda < \text{Re} z < \beta \). Then for \( \lambda < \gamma \leq \beta \), and any \( q > 0 \), we have
provided the right hand side is finite.

Apart from this we have to use a well-known theorem of Montgomery and Vaughan. For reference see for instance my paper [4], where I give a simple proof of a weaker result which is sufficient for the purposes of this paper.

We now split up the proof of theorem 2 into several steps and give a brief sketch of these steps.

§ 2. Proof of Theorem 2

Step 1. Let

\[ I(\sigma) = \frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} |\phi(s)|^{2/k} \right. \]

\[ \left. \cdot \exp \left( \left( s - it_0 \right)^{4a+2} \right) dt \right) dt_0, \]

and assume that \( I(\vartheta) < V(\vartheta) H^{1-2\vartheta} \). The constant \( a \) shall be a sufficiently large positive integer. As already stated we set

\[ X = [2 - A - 100 H] + 2, \text{ and} \]

\[ V(\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n|^2 \left( \frac{H}{\mu_n} \right)^{2\sigma} \text{ and} \]

we impose \( \vartheta < \beta < \vartheta_1 < \vartheta_2 < \vartheta_3 < \vartheta + \frac{1}{2} \).
Step 2. Next we write

$$J(\sigma) = \frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) - P_k(s) \right|^2 \right) \left| \text{Exp} \left( (s-it_0)^{4a+2} \right) \right| dt \ dt_0$$

where $P(s) = \prod_{n} b_n \mu_n^{-s}$. It is easily seen that

$$\Delta(X) = O(X^B)$$

and also $1 + O(X^{-B})$ where $B > 0$ is an arbitrary constant and the $O$-constant depends only on $B$ and $a$.

Again $a_n = \sum_{1}^{n} \mu_n$, $\sum_{1}^{n} (b_n b_{n+1} \ldots b_{n+k})$ and for all $N \geq 1$, we have

$$\sum_{1}^{N} \frac{1}{\mu_n \ldots \mu_k} = O(N^k).$$

From these remarks it is clear that $\phi(s) - P_k(s)$ decays fast enough to ensure $J(\sigma) < 1$ when $\sigma$ is large enough. Now from an easy application of a theorem of Gabriel (Theorem 3 above) it follows that in $\sigma \geq \alpha$, $J(\sigma)$ is $\ll_{\varepsilon} [J(\alpha)]^{1-\varepsilon}$ for every positive constant $\varepsilon$ uniformly in $\sigma$, and so in $(\sigma \geq \beta, \ T \leq t \leq T+H)$, $|\phi(s)|$ is bounded above by a constant power of $H$. (Here for getting the last bound we have to use the fact that for any analytic function $\phi(s)$, $|\phi(s)|^{2/k}$ is bounded by its mean value over a disc of (positive but sufficiently small) constant radius with $s$ as centre).

Step 3. An easy application of a well-known Montgomery–Vaughan theorem (refer [4] for instance) shows that

$$\frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} \left| P(s) \right|^2 \left| \text{Exp} \left[ (s-it_0)^{4a+2} \right] \right| dt \right) dt_0 \quad = O(\sqrt{\psi(\sigma)} H^{1-2\sigma}).$$
From this and the estimate \( J(\sigma) \ll \varepsilon (J(\alpha))^{1-\varepsilon} \) it follows that

\[
A^a \int_\mathcal{A} \left( \frac{1}{H} \int_T^T \left( \int_{-\infty}^{\infty} (|\phi(s) - P(s)|^2 + r_0(s)^2) \right) dt \right) dt_0 \right) d\sigma
= O \left( (J(\alpha))^{1-\varepsilon} + V(\alpha) H^{1-2\alpha + \varepsilon} \right).
\]

Note that \( V(\sigma) \) and \( V^*(\sigma) H^{1-2\sigma} \) are respectively monotonic increasing and monotonic decreasing functions of \( \sigma \), where \( V^*(\sigma) \) is the same as \( V(\sigma) \) with the terms \( \mu_n \ll 1 \) omitted.

From now on we assume that \( V(\alpha) \) is bounded below by a constant negative power of \( H \). Under this assumption it follows that the integral just considered is

\[
O(\varepsilon) (V(\alpha) H^{1-2\alpha + \varepsilon}) \text{ for every positive constant } \varepsilon.
\]

Hence there exist intervals \( I_1 \) and \( I_2 \) contained in \((T, T + H/10)\) and \((T + H - H/10, T + H)\) respectively, for which the lengths are \( 4H^\delta \) (\( \delta \) being any fixed constant satisfying \( 0 < \delta < \frac{1}{100} \)) each and further \( I(I_1, \alpha) \)

\[
= \int_\mathcal{A} \left( \frac{1}{H} \int_{I_1}^T \int_{-\infty}^{\infty} (|\phi(s) - P(s)|^2 + r_0(s)^2) \right) dt dt_0 \right) d\sigma
\]

and \( I(I_2, \alpha) \) defined similarly (by replacing \( I_1 \) by \( I_2 \)) satisfy

\[
l(I_1, \alpha) + I(I_2, \alpha) = O(V(\alpha) H^{1-2\alpha + \varepsilon}).
\]
Hence by the principle for the mean value over discs referred to in the second step, we see that in \( \beta \leq \sigma \leq A^2 \), \( t \) in any of the intervals \( I_1, I_2 \) we have
\[
|\phi(s)|^{2/k} = O \left( V(\omega) H^{-2} d + 2 E + \delta \right).
\]

**Step 4.** Let \( H_1 \) and \( H_2 \) be the mid points of \( I_1 \) and \( I_2 \) respectively. We now obtain a lower bound for at least one of the mean-values \( K(\omega_1) \) or \( K(\omega_2) \) where \( K(\omega) \) is defined by
\[
K(\omega) = \frac{1}{H_2 - H_1} \int_{H_1}^{H_2} |F_0(s)|^2 \, dt, \quad (\beta \leq \sigma \leq A^2).
\]

Note that when we replace \( H_1 \) and \( H_2 \) by other points in \( I_1 \) and \( I_2 \) the mean value \( K(\omega) \) changes by an amount which is at most \( O(E) \) where \( E = V(\omega) H^{-2} d + 2 E + 2 \delta \). Hence if \( \sigma \) denotes any of \( \omega_2 \) or \( \omega_3 \) and \( H_1 < t < H_2 \), we see that if \( j \)

\[
\frac{j!}{2\pi i} \int_{L_0} \frac{F_0(s + w)(2X)^w}{w(w + 1) \ldots (w + j)} \, dw = \sum_{\mu_n \leq 2X} b_n \mu^{-s} \left( 1 - \frac{\mu_n}{2X} \right)^j,
\]

where \( L_0 \) is the line \( \text{Re} \, w = A^2 \). Deform the line \( L_0 \) to the contour described by the lines \( L_1, L_2, L_3, L_4, L_5 \) (in this order) defined as follows. Let \( H_3 = H_1 - H \delta \), \( H_4 = H_2 + H \delta \) where \( \delta \) is a small positive constant. \( L_1 \) and
L \_5 are the portions $\text{Im} \, w \leq -H_3$ and $\text{Im} \, w \geq H_4$ respectively of $L_0$. $L_2$ is the line segment

\[
( \text{Im} \, w = -H_3, \quad \alpha_1 - \sigma \leq \text{Re} \, w \leq A^2 )
\]

and $L_4$ is the line segment

\[
( \text{Im} \, w = H_4, \quad \alpha_1 - \sigma \leq \text{Re} \, w \leq A^2 )
\]

$L_3$ is the line segment ($\text{Re} \, w = \alpha_1 - \sigma, -H_3 \leq \text{Im} \, w < H_4$).

Taking the mean square after deformation of $L_0$ we find from the equation (1), (Note that the only pole to be taken care of is $w = 0$),

\[
\begin{align*}
&K(\sigma) \ll V_1(\sigma) H^{1-2\sigma} + (K(\alpha_1) + E) H^{-2}(\sigma - \alpha_1) \\
\text{(where } V_1(\sigma) \text{ is defined below) and also} \\
&V(\sigma) H^{1-2\sigma} \ll K(\sigma) + (K(\alpha_1) + E) H^{-2}(\sigma - \alpha_1).
\end{align*}
\]

The reason for this is that the mean square of the RHS of (1) is $\gg$ and $\ll V_1(\sigma) H^{1-2\sigma}$ where $V_1(\sigma)$ is defined by

\[
V_1(\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} \left| b_n \right|^2 \left( -\frac{\mu}{2X} \right)^2 j \left( \frac{H}{\mu} \right)^{2\sigma}.
\]

Since $V(\alpha_1) \ll V(\sigma)$ we may omit the term containing $E$ in the second of the equations (2), provided $\alpha_1 - \alpha < \frac{1}{\sigma}$ (which is true because of our assumptions). This gives us

\[
V(\sigma) H^{1-2\sigma} \ll K(\sigma) + K(\alpha_1) H^{-2}(\sigma - \alpha_1); \quad (\sigma = \alpha_2, \alpha_3)
\]

If we put $\sigma = \alpha_2$ we get a lower bound for one at least of the quantities $K(\alpha_1)$ or $K(\alpha_2)$.

We now deduce from the last inequality

\[
V(\sigma) H^{1-2\sigma} \ll I(\sigma) + I(\alpha_1) H^{-2}(\sigma - \alpha_1),
\]

\[
(\sigma = \alpha_2, \alpha_3).
\]
This is possible since in the range \((\sigma > a_1, H_3 < t < H_4)\),
\(\|F_0 (s)\|_2^{2k}\) is a very good approximation (in the mean) to
\(\|\phi (s)\|_2^{2/k}\) and we leave the details as an exercise. From the
last inequality it follows that \(I(\sigma) \gg V(\sigma) H^{1 - 2\sigma}\), for one
at least of the values \(\sigma = a_1\) or \(a_2\).

Next by the convexity theorem of Gabriel (Theorem 3),
we find that with the value of \(\sigma (a_1\) or \(a_2\) determined),

\[
(3) \quad (I(\sigma)) \leq (1(a_3)) \quad (1(a_3)) \quad (1(a_3))
\]

Moreover by the arguments used in the first of the
inequalities in (2) we get (by taking \(X\) in place of \(2X\) in (1)).

\[
I(a_3) \ll V(a_3) H^{1 - 2a_3} + (I(\sigma) + E) H^{-2(a_3 - \sigma)} + E.
\]

Now \(E = V(a) H^{-2a + 2e + 2%} < V(a_3) H^{1 - 2a_3}\)

\((\text{since by our assumptions, } a_3 - a < \frac{1}{2})\) by a small choice
of the positive constants \(e, \delta\). Thus we get

\[
I(a_3) \ll V(a_3) H^{1 - 2a_3} + I(\sigma) H^{-2(a_3 - \sigma)},
\]

and so by (3)

\[
(4) \quad (I(\sigma)) \leq (1(a_3)) \quad (V(a_3) H^{1 - 2a_3} + I(\sigma) H^{-2(a_3 - \sigma)})^{\sigma - a}.
\]

This holds for either \(\sigma = a_1\) or \(\sigma = a_2\) and gives us
Either
\[
I(\sigma) \gg V(\sigma) H^{1-2\sigma} > V(\sigma) H^{1-2\sigma},
\]
\[
\mathcal{d}_3 - \sigma
\]
or
\[
I(\sigma) \gg (V(\sigma) H^{1-2\sigma}) (V(\sigma) H^{1-2\sigma} - \sigma).
\]

The second of these inequalities gives

\[
I(\sigma) \gg H^{1-2\sigma} V(\sigma) \left\{ \frac{V(\sigma)}{V(\sigma_3)} \right\}
\]

Since \( V(\sigma) < V(\sigma_1) < V(\sigma) < V(\sigma_3) \) and since \( \frac{u - \sigma}{\sigma_3 - u} \) is an increasing function of \( u \) in \( \beta < u < \sigma_3 \), we get finally

\[
I(\sigma) \gg V(\sigma) H^{1-2\sigma} \left\{ \frac{V(\sigma_1)}{V(\sigma_3)} \right\}
\]

**Step 5.** Step 4 nearly completes the proof. For we could have started with a slight modification of \( I(\sigma) \) by averaging over a slightly smaller interval contained in \( (T, T + H) \) instead of \( (T, T + H) \). For instance by cutting off bits of length \( H^8 \) on either side. The decaying factor \( \exp\left( (s - i\sigma_0)^{4a + 2} \right) \) enables us to replace the modified

\[
I(\sigma) \text{ by } \frac{1}{H} \int_T^{T+H} |\phi(t)|^{2/k} dt
\]
in the last lower bound.

Steps 1, 2, 3, 4 and 5 complete the proof of theorem 2.
Theorem 4

With the notation of theorem 2, we have,

\[ \frac{1}{H} \int_{T}^{T+H} |\phi (\alpha + it)|^2 \, dt \gg \]

\[ \left( V(\alpha)H^{-1} - 2\alpha \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\} \frac{\alpha_2 - \alpha}{\alpha_3 - \alpha} \right)^k \]

where the constant implied by the Vinogradov symbol is effective.

Remark. This theorem will be used in [1].

References


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