Progress Towards a conjecture on the mean-value of titchmarsh series-II
K Ramachandra

To cite this version:
K Ramachandra. Progress Towards a conjecture on the mean-value of titchmarsh series-II. Hardy-Ramanujan Journal, Hardy-Ramanujan Society, 1981, 4, pp.1 - 12. <hal-01103883>

HAL Id: hal-01103883
https://hal.archives-ouvertes.fr/hal-01103883
Submitted on 15 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
PROGRESS TOWARDS A CONJECTURE ON THE MEAN-VALUE OF TITCHMARSH SERIES-II

By K. RAMACHANDRA

§ 1. Introduction

The result of this paper may be considered as complementary to that of my earlier paper [2], on Titchmarsh series. Although not as interesting as the earlier result, the result of the present paper finds a nice application, (See [1]). In [3] I defined a class of series called Titchmarsh series and I now start by recalling its definition.

Titchmarsh Series. (or briefly K D T series).

Let $A > 10$ be a constant.

Let $\frac{1}{A} < \mu_1 < \mu_2 < \ldots$ where

$$\frac{1}{A} < \mu_{n+1} - \mu_n < A \text{ (for } n = 1, 2, 3, \ldots).$$

(In [2] the notation is slightly different and we have used there $\lambda_n$ instead of $\mu_n$ and for simplicity assumed $\lambda_1 = 1$. Also we have used there $a_1, a_2, \ldots$ in place of our present $b_1, b_2, \ldots$ and assumed for simplicity $a_1 = 1$. We have written there $F(s)$ instead of $F_0(s)$). Let $b_1, b_2, \ldots$ be a sequence of complex numbers possibly depending on a parameter $H > 10$ such that

$$|b_n| < (\mu_n H)^A.$$ Put $F_0(s) = \sum_{n=1}^{\infty} b_n \mu_n - s$ where $s = \sigma + it$. 
If $F_0(s)$ is called a KDT series if there exists a constant $A > 10$ and a system of rectangles $R(T, T+H)$ defined by $\{ \sigma > 0, T < t < T+H \}$ where $10 < H < T$, and $T$ (which may be related to $H$) tends to infinity, and $F_0(s)$ admits an analytic continuation into these rectangles and the maximum of $|F_0(s)|$ taken over $R(T, T+H)$ does not exceed $\text{Exp}(H^A)$.

I then made the following conjecture:

**Conjecture**

For a KDT series $F_0(s)$, we have,

$$\frac{1}{H} \int_{L}^{R} |F(it)|^2 dt > C_A \sum_{n} \frac{\mu_n}{X} |b_n|^2,$$

where $X = 2 + D_A H$, $L$ denotes the side ($\sigma = 0$, $T < t < T + H$) of $R(T, T+H)$, and $C_A$ and $D_A$ are positive constants depending only on $A$, provided $\mu_1 = b_1 = 1$.

I proved the following theorem.

**Theorem 1**

Under the restrictions $\mu_1 = b_1 = 1$, we have,

$$\frac{1}{H} \int_{L}^{R} |F_0(it)|^2 dt > C_A \sum_{n} \frac{\mu_n}{X} |b_n|^2 = \left(1 - \frac{\log \mu n}{\log H} + \frac{1}{\log \log H}\right),$$

where $X = 2 + D_A H$ and, $C_A$ and $D_A$ are effective positive constants depending only on $A$.

I now prove the following theorem.
Theorem 2

For some convenience let us assume in the definition of Titchmarsh series \( F_0 (s) \) the rectangles \( R (T, T + H) \) to be \((\sigma > \beta, T < t < T + H)\) where \( \beta \) is a positive constant such that \( 0 < \beta < \alpha_1 < \frac{1}{2} \), where \( \alpha_1 \) is another constant. Let \( k \geq 2 \) be an integer and write \( F (s) = (F_0 (s))^{k} = \lambda \sum_{n = 1}^{\infty} n^{-s} (a_n \lambda_n^{s}) \), a series which is surely convergent where \( F_0 (s) \) is absolutely convergent. Put \( Y = (M + H) \) where

\[
100 \quad -10
\]

\[
\lambda = kA (\alpha_1 - \beta) \quad \text{and} \quad M = \max |F_0 (s)| \quad \text{taken over } R (T, T + H).
\]

Define the entire function \( \phi (s) \) by

\[
\phi (s) = \lambda \sum_{n = 1}^{\infty} a_n \lambda_n^{-s} \Delta \left( \frac{Y}{\lambda_n} \right)
\]

where for \( X > 0 \), \( \Delta (X) \) is defined by

\[
\Delta (X) = \frac{1}{2 \pi i} \int_{2 - i \infty}^{2 + i \infty} X^W \exp \left( \frac{4a + 2}{W} \right) \frac{dW}{W},
\]

\( a \) being a suitable positive integer constant at our choice. We now suppose that \( \alpha, \beta, \alpha_1, \alpha_2, \alpha_3 \) are constants satisfying

\[
\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}.
\]

Put \( X = \left[ 2 - A - 100 \right] H + 2 \) and

\[
V (\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n| \left( \frac{H}{\mu_n} \right)^{2 \sigma}.
\]
Then we have,

\[ \frac{1}{H} \int I \phi (a + it) \frac{2^k}{L} dt \]

\[ \gg V(a) H^{1-2a} \left\{ \frac{V(a_1)}{V(a_3)} \right\}, \]

provided only that \( V(a) \neq 0 \) and \( (\log V(a)) \left( \log H \right)^{-1} \) is bounded below by a negative constant. Here the constant implied by the Vinogradov symbol \( \gg \) depends only on \( A, k, a, \beta, a_1, a_2, a_3 \) and the negative constant referred to just now. Further it is effective.

\textbf{Remark:} It will be clear from the proof that if

\[ \mu_n = n (n = 1, 2, 3, \ldots) \]

then we can choose \( X = \left[ \frac{H}{100} \right] + 2. \)

The object of this paper is to prove theorem 2. The proof of the theorem is fairly long. The proof depends upon a special case of a convexity theorem of R. M. Gabriel which we state below (in the notation of D. R. Heath–Brown’s paper [2]; for the more general theorem of Gabriel see the reference in [2] or Titchmarsh’s famous book [5] pages 203 and 337.)

\textbf{Theorem 3}

Let \( f(z) \) be regular in the infinite strip \( \alpha < \Re z < \beta \) and continuous for \( \alpha < \Re z < \beta \). Suppose \( f(z) \to 0 \) as \( \Im z \to \infty \), uniformly in \( \alpha < \Re z < \beta \). Then for \( \alpha < \gamma < \beta \), and any \( q > 0 \), we have
\[
\int_{-\infty}^{\infty} \left| f(\gamma + it) \right|^q \, dt \leq \left( \int_{-\infty}^{\infty} \left| f(\alpha + it) \right|^q \, dt \right)^{\frac{\beta - \gamma}{\beta - \alpha}} \cdot \left( \int_{-\infty}^{\infty} \left| f(\beta + it) \right|^q \, dt \right)^{\frac{\gamma - \alpha}{\beta - \alpha}},
\]

provided the right hand side is finite.

Apart from this we have to use a well-known theorem of Montgomery and Vaughan. For reference see for instance my paper [4], where I give a simple proof of a weaker result which is sufficient for the purposes of this paper.

We now split up the proof of theorem 2 into several steps and give a brief sketch of these steps.

§ 2. Proof of Theorem 2

Step 1. Let

\[
I(\sigma) = \frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) \right|^{2/k} \right) \exp \left( \left( s - it_0 \right)^{4a+2} \right) \, dt \, dt_0,
\]

and assume that \( I(\alpha) < V(\alpha) H^{1-2\alpha} \). The constant \( a \) shall be a sufficiently large positive integer. As already stated we set

\[
X = \left[ 2^{-A-100} H \right] + 2,
\]

and

\[
V(\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} \left| b_n \right|^2 \left( \frac{H}{\mu_n} \right)^{2\sigma},
\]

we impose \( \alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2} \).
Step 2. Next we write
\[
J(\sigma) = \frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) - P_k(s) \right| \right)^{2/k} dt_0
\]
where \( P(s) = \sum b_n \mu_n^{-s} \). It is easily seen that \( \Delta(X) = O(X^B) \) and also \( 1 + O(X^{-B}) \) where \( B > 0 \) is an arbitrary constant and the \( O \)-constant depends only on \( B \) and \( a \).

Again \( a_n = \mu_1 \cdots \mu_k = \lambda_n (b_n \cdots b_n) \) and for all \( N \geq 1 \), we have \( \sum_{N < \mu_n} 1 = O(N^k) \). From these remarks it is clear that \( \phi(s) - P_k(s) \) decays fast enough to ensure \( J(\sigma) < 1 \) when \( \sigma \) is large enough. Now from an easy application of a theorem of Gabriel (Theorem 3 above) it follows that in \( \sigma > \delta \), \( J(\sigma) \ll \varepsilon \left[ J(\delta) \right]^{1-\varepsilon} \) for every positive constant \( \varepsilon \) uniformly in \( \sigma \), and so in \( (\sigma > \beta, T < t < T + H) \), \( |\phi(s)| \) is bounded above by a constant power of \( H \). (Here for getting the last bound we have to use the fact that for any analytic function \( \phi(s) \), \( |\phi(s)|^{2/k} \) is bounded by its mean value over a disc of (positive but sufficiently small) constant radius with \( s \) as centre).

Step 3. An easy application of a well-known Montgomery–Vaughan theorem (refer [4] for instance) shows that
\[
\frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} \left| P(s) \right|^2 \text{Exp} \left[ (s-it_0)^{4a+2} \right] \right) dt_0 = O(V(\sigma) H^{1-2\sigma}).
\]
From this and the estimate $J(\sigma) \ll \varepsilon (J(\alpha))^{1-\varepsilon}$ it follows that

$$
A^g \int_{\mathcal{A}} \left( \int_{T}^{T+H} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (f(s) - P(s)) \frac{2}{k} \right. \right.
$$

$$
\left. + |P(s)|^2 \right) \exp \left( (s-it_0)^{4a+2} \right) dt \ dt_0 \right) d\sigma
$$

$$
= O \left( (J(\alpha))^{1-\varepsilon} + V(\alpha) H^{1-2\alpha} \right).
$$

Note that $V(\sigma)$ and $V^*(\sigma) H^{1-2\sigma}$ are respectively monotonic increasing and monotonic decreasing functions of $\sigma$, where $V^*(\sigma)$ is the same as $V(\sigma)$ with the terms $\mu_n \ll 1$ omitted.

From now on we assume that $V(\alpha)$ is bounded below by a constant negative power of $H$. Under this assumption it follows that the integral just considered is

$$
O_\varepsilon (V(\alpha) H^{1-2\alpha + \varepsilon}) \quad \text{for every positive constant } \varepsilon.
$$

Hence there exist intervals $I_1$ and $I_2$ contained in

$$
\left( T, T + \frac{H}{10} \right) \quad \text{and} \quad \left( T + H - \frac{H}{10}, T + H \right)
$$

respectively, for which the lengths are $4 H^\varepsilon$ ($\varepsilon$ being any fixed constant satisfying $0 < \varepsilon < \frac{1}{100}$) each, and further $I(I_1, \alpha)$

$$
= \int_{I_1} \left( \int_{T}^{T+H} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (f(s) - P(s)) \frac{2}{k} \right. \right.
$$

$$
\left. + |P(s)|^2 \right) \exp \left( (s-it_0)^{4a+2} \right) dt \ dt_0 \right) d\sigma
$$

and $I(I_2, \alpha)$ defined similarly (by replacing $I_1$ by $I_2$) satisfy

$$
I(I_1, \alpha) + I(I_2, \alpha) = O(V(\alpha) H^{1-2\alpha + \varepsilon}).
$$
Hence by the principle for the mean value over discs referred to in the second step, we see that in \((\beta \leq \sigma \leq A^2, \quad \mathcal{I} \text{ in any of the intervals } I_1, I_2)\) we have

\[
|\phi(s)|^{2/k} = O\left(V(\mathcal{d}) H^{-2} d + 2 \varepsilon + \delta \right).
\]

**Step 4.** Let \(H_1\) and \(H_2\) be the mid points of \(I_1\) and \(I_2\) respectively. We now obtain a lower bound for at least one of the mean-values \(K(\mathcal{d}_1)\) or \(K(\mathcal{d}_2)\) where \(K(\sigma)\) is defined by

\[
K(\sigma) = \frac{1}{H_2 - H_1} \int_{H_1}^{H_2} |F_0(s)|^2 \, dt, \quad (\beta < \sigma < A^2).
\]

Note that when we replace \(H_1\) and \(H_2\) by other points in \(I_1\) and \(I_2\) the mean value \(K(\sigma)\) changes by an amount which is at most \(O(\varepsilon)\) where \(\varepsilon = V(\mathcal{d}) H^{-2} d + 2 \varepsilon + 2 \delta\). Hence if \(\sigma\) denotes any of \(\mathcal{d}_2\) or \(\mathcal{d}_3\) and \(H_1 < t < H_2\), we see that if \(j\) is a large positive integer constant,

\[
(1) \quad \frac{j!}{2\pi i} \int_{L_0} \frac{F_0(s+w)(2X)^w}{w(w+1) \ldots (w+j)} \, dw = \sum_{\mu_n < 2X} b_n \mu^{-s} \left(1 - \frac{\mu_n}{2X}\right)^{j},
\]

where \(L_0\) is the line \(\text{Re} w = A^2\). Deform the line \(L_0\) to the contour described by the lines \(L_1, L_2, L_3, L_4, L_5\) (in this order) defined as follows. Let \(H_3 = H_1 - H^\delta\), \(H_4 = H_2 + H^\delta\) where \(\delta\) is a small positive constant. \(L_1\) and
$L_5$ are the portions $\text{Im } w \leq -H_3$ and $\text{Im } w \geq H_4$ respectively of $L_0$. $L_2$ is the line segment
\[
(L \text{ Im } w = -H_3, \quad \alpha_1 - \sigma \leq \text{Re } w \leq A^2)
\]
and $L_4$ is the line segment
\[
(L \text{ Im } w = H_4, \quad \alpha_1 - \sigma \leq \text{Re } w \leq A^2).
\]
$L_3$ is the line segment $(\text{Re } w = \alpha_1 - \sigma, \quad -H_3 \leq \text{Im } w \leq H_4)$. Taking the mean square after deformation of $L_0$ we find from the equation (1) (Note that the only pole to be taken care of is $w = 0$),
\[
\begin{align*}
&K (\sigma) \ll V_1 (\sigma) H^{1-2\sigma} + (K (\alpha_1) + E) H^{-2(\sigma - \alpha_1)} \\
&\text{where } V_1 (\sigma) \text{ is defined below) and also}
\end{align*}
\]
\[
\begin{align*}
&V (\sigma) H^{1-2\sigma} \ll K (\sigma) + (K (\alpha_1) + E) H^{-2(\sigma - \alpha_1)}.
\end{align*}
\]
The reason for this is that the mean square of the RHS of (1) is $\gg$ and $\ll V_1 (\sigma) H^{1-2\sigma}$ where $V_1 (\sigma)$ is defined by
\[
V_1 (\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} |b_n| \left( -\tfrac{\mu}{2} \right)^2 \left( \frac{H}{\mu} \right)^2 \sigma.
\]
Since $V (\alpha) \ll V (\sigma)$ we may omit the term containing $E$ in the second of the equations (2), provided $\alpha_1 - \sigma < \frac{1}{4}$ (which is true because of our assumptions). This gives us
\[
V (\sigma) H^{1-2\sigma} \ll K (\sigma) + K (\alpha_1) H^{-2(\sigma - \alpha_1)}; (\sigma = \alpha_2, \alpha_3)
\]
If we put $\sigma = \alpha_2$ we get a lower bound for one at least of the quantities $K (\alpha_1)$ or $K (\alpha_2)$.

We now deduce from the last inequality
\[
V (\sigma) H^{1-2\sigma} \ll I (\sigma) + I (\alpha_1) H^{-2(\sigma - \alpha_1)},
\]
\[
(\sigma = \alpha_2, \quad \alpha_3).
\]
This is possible since in the range \( \sigma > \ell_1, H_3 < t < H_4 \),
\(|F_0(s)|^2\) is a very good approximation (in the mean) to
\(|\phi(s)|^{2/k}\) and we leave the details as an exercise. From the
last inequality it follows that \( I(\sigma) \gg V(\sigma) H^{1-2\sigma} \), for one
at least of the values \( \sigma = \ell_1 \) or \( \ell_2 \).

Next by the convexity theorem of Gabriel (Theorem 3),
we find that with the value of \( \sigma (\ell_1 \) or \( \ell_2 \) determined),

\[
(3) \quad \frac{d_3 - d}{(I(\sigma))} < \left( \frac{d_3 - \sigma}{(I(d))} \right) \left( \frac{\sigma - d}{(I(d_3))} \right).
\]

Moreover by the arguments used in the first of the
inequalities in (2) we get (by taking \( X \) in place of \( 2X \) in (1)).

\[
I(d_3) \ll V(d_3) H^{1-2d_3} + (I(\sigma) + \epsilon) H^{-2(d_3 - \sigma)} + \epsilon.
\]

Now \( \epsilon = V(d) H^{-2d + 2\epsilon + 2\delta} < V(d_3) H^{1 - 2d_3}\)

(since by our assumptions, \( d_3 - d < \frac{1}{2} \)) by a small choice
of the positive constants \( \epsilon, \delta \). Thus we get

\[
I(d_3) \ll V(d_3) H^{1-2d_3} + I(\sigma) H^{-2(d_3 - \sigma)},
\]

and so by (3)

\[
(4) \quad \frac{d_3 - d}{(I(\sigma))} < \left( \frac{d_3 - \sigma}{(I(d))} \right) \left( V(d_3) H^{1-2d_3} + I(\sigma) H^{-2(d_3 - \sigma)} \right)^{\sigma - d}.
\]

This holds for either \( \sigma = \ell_1 \) or \( \sigma = \ell_2 \) and gives us
The second of these inequalities gives

\[
I(\sigma) \gg H^{1-2\sigma} V(\sigma) \left\{ \frac{V(\sigma)}{V(\sigma_3)} \right\}
\]

Since \( V(\sigma) \leq V(\sigma_1) \leq V(\sigma) \leq V(\sigma_3) \) and since \( \frac{u-\sigma}{\sigma_3-u} \) is an increasing function of \( u \) in \( \beta \leq u < \sigma_3 \), we get finally

\[
I(\sigma) \gg V(\sigma) H^{1-2\sigma} \left\{ \frac{V(\sigma_1)}{V(\sigma_3)} \right\}
\]

**Step 5.** Step 4 nearly completes the proof. For we could have started with a slight modification of \( I(\sigma) \) by averaging over a slightly smaller interval contained in \( (T, T+H) \) instead of \( (T, T+H) \). For instance by cutting off bits of length \( H^8 \) on either side. The decaying factor \( \text{Exp} \left( (s - it_0) 4a + 2 \right) \) enables us to replace the modified

\[
I(\sigma) \text{ by } \frac{1}{H} \int_T^{T+H} |\phi(s)|^{2/k} |s|^{2/k} \ dt
\]

in the last lower bound.

Steps 1, 2, 3, 4 and 5 complete the proof of theorem 2.
Theorem 4

With the notation of theorem 2, we have,

\[
\frac{1}{H} \int_T^{T+H} |\phi(\alpha + it)|^2 \, dt \gg \left( V(\alpha)H^1 - 2\alpha \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\} \frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2} \right)^k
\]

where the constant implied by the Vinogradov symbol is effective.

Remark. This theorem will be used in [1].

References


Manuscript Completed In The Final Form On 11-Oct-1980

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Colaba, Bombay 400 005
(India).