Hardy-Ramanujan Journal
Vol. 4 (1981) 1–12

PROGRESS TOWARDS A CONJECTURE ON THE MEAN-VALUE OF TITCHMARSH SERIES-II

By K. RAMACHANDRA

§ 1. Introduction

The result of this paper may be considered as complementary to that of my earlier paper [2], on Titchmarsh series. Although not as interesting as the earlier result, the result of the present paper finds a nice application, (See [1]). In [3] I defined a class of series called Titchmarsh series and I now start by recalling its definition.

Titchmarsh Series. (or briefly K D T series).

Let $A > 10$ be a constant.

Let $\frac{1}{A} < \mu_1 < \mu_2 < \ldots$ where

$$\frac{1}{A} < \mu_{n+1} - \mu_n < A \text{ (for } n = 1,2,3, \ldots).$$

(In [2] the notation is slightly different and we have used there λ_n instead of μ_n and for simplicity assumed $\lambda_1 = 1$. Also we have used there $a_1, a_2, \ldots$ in place of our present $b_1, b_2 \ldots$ and assumed for simplicity $a_1 = 1$. We have written there $F(s)$ instead of $F_0(s)$). Let $b_1, b_2, \ldots$ be a sequence of complex numbers possibly depending on a parameter $H > 10$ such that $|b_n| < (\mu_n H)^A$. Put $F_0(s) = \sum_{n=1}^{\infty} b_n \mu_n^{-s}$ where $s = \sigma + it$.

$F_0(s)$ is called a KDT series if there exists a constant $A > 10$ and a system of rectangles $R(T, T+H)$ defined by $\{\sigma > 0, T < t < T+H\}$ where $10 < H < T$, and T (which may be related to H) tends to infinity, and $F_0(s)$ admits an analytic continuation into these rectangles and the maximum of $|F_0(s)|$ taken over $R(T, T+H)$ does not exceed $\operatorname{Exp}(H^A)$.

I then made the following conjecture

Conjecture

For a KDT *series* $F_0(s)$, *we have,*

$$\frac{1}{H} \int_L |F_0(it)|^2 \, dt > C_A \sum_{\mu_n < X} |b_n|^2,$$

where $X = 2 + D_A H$, L *denotes the side* ($\sigma = 0, T < t < T+H$) *of* $R(T, T+H)$, *and* C_A *and* D_A *are positive constants depending only on* A, *provided* $\mu_1 = b_1 = 1$.

I proved the following theorem.

Theorem 1

Under the restrictions $\mu_1 = b_1 = 1$, *we have,*

$$\frac{1}{H} \int_L |F_0(it)|^2 \, dt > C_A \sum_{\mu_n < X} |b_n|^2 \left(1 - \frac{\log \mu_n}{\log H} + \frac{1}{\log \log H}\right),$$

where $X = 2 + D_A H$ *and,* C_A *and* D_A *are effective positive constants depending only on* A.

I now prove the following theorem.

Theorem 2

For some convenience let us assume in the definition of Titchmarsh series $F_0(s)$ *the rectangles* $R(T, T+H)$ *to be* $(\sigma > \beta, T \leqslant t \leqslant T+H)$ *where* β *is a positive constant such that* $0 < \beta < \alpha_1 < \frac{1}{2}$, *where* α_1 *is another constant. Let* $k \geqslant 2$ *be an integer and write* $F(s) = (F_0(s))^k = \sum_{n=1}^{\infty} (a_n \lambda_n^{-s})$, *a series which is surely convergent where* $F_0(s)$ *is absolutely convergent. Put* $Y = (M+H)^{\lambda}$ *where* $\lambda = kA^{100} (\alpha_1 - \beta)^{-10}$ *and* $M =$ *maximum of* $|F_0(s)|$ *taken over* $R(T, T+H)$. *Define the entire function* $\phi(s)$ *by*

$$\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta\left(\frac{Y}{\lambda_n}\right)$$

where for $X > 0$, $\Delta(X)$ *is defined by*

$$\Delta(X) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} X^{W} \, Exp\,(W^{4a+2}) \frac{dW}{W},$$

a being a suitable positive integer constant at our choice. We now suppose that $\alpha, \beta, \alpha_1, \alpha_2, \alpha_3$ *are constants satisfying*

$$\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}.$$

Put $X = [2^{-A-100} H] + 2$ *and*

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n \leqslant X} |b_n|^2 \left(\frac{H}{\mu_n}\right)^{2\sigma}.$$

Then we have,

$$\frac{1}{H}\int_{L} |\phi(\alpha+it)|^{2k^{-1}}\, dt$$

$$\gg V(\alpha)\, H^{1-2\alpha} \left\{\frac{V(\alpha_1)}{V(\alpha_3)}\right\}^{\frac{\alpha_2-\alpha}{\alpha_3-\alpha_2}},$$

provided only that $V(\alpha) \neq 0$ *and* $(\log V(\alpha))\,(\log H)^{-1}$ *is bounded below by a negative constant. Here the constant implied by the Vinogradov symbol* $\gg$ *depends only on* A, k, α, β, α_1, α_2, α_3 *and the negative constant referred to just now. Further it is effective.*

Remark : It will be clear from the proof that if

$\mu_n = n$ $(n = 1,2,3,\ldots)$ then we can choose $X = \left[\frac{H}{100}\right] + 2.$

The object of this paper is to prove theorem 2. The proof of the theorem is fairly long. The proof depends upon a special case of a convexity theorem of R. M. Gabriel which we state below (in the notation of D. R. Heath-Brown's paper [2] ; for the more general theorem of Gabriel see the reference in [2] or Titchmarsh's famous book [5] pages 203 and 337.)

Theorem 3

Let $f(z)$ *be regular in the infinite strip* $\alpha < \mathrm{Re}\, z < \beta$ *and continuous for* $\alpha \leq \mathrm{Re}\, z \leq \beta$. *Suppose* $f(z) \to 0$ *as* $|\mathrm{Im}\, z| \to \infty$, *uniformly in* $\alpha \leq \mathrm{Re}\, z \leq \beta$. *Then for* $\alpha \leq \gamma \leq \beta$, *and any* $q > 0$, *we have*

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^{q}\, dt < \left(\int_{-\infty}^{\infty} |f(\alpha + it)|^{q}\, dt\right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_{-\infty}^{\infty} |f(\beta + it)|^{q}\, dt\right)^{\frac{\gamma-\alpha}{\beta-\alpha}},$$

provided the right hand side is finite.

Apart from this we have to use a well-known theorem of Montgomery and Vaughan. For reference see for instance my paper [4], where I give a simple proof of a weaker result which is sufficient for the purposes of this paper.

We now split up the proof of theorem 2 into several steps and give a brief sketch of these steps.

§ 2. Proof of Theorem 2

Step I. *Let*

$$I(\sigma) = \frac{1}{H}\int_{T}^{T+H} \left(\int_{-\infty}^{\infty} |\phi(s)|^{2/k} \,|\, \mathrm{Exp}\,((s - it_0)^{4a+2}) \,|\, dt\right) dt_0,$$

and assume that $I(\alpha) < V(\alpha)\, H^{1-2\alpha}$. The constant a shall be a sufficiently large positive integer. As already stated we set

$$X = [2^{-A-100} H] + 2, \text{ and}$$

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n|^2 \left(\frac{H}{\mu_n}\right)^{2\sigma} \quad \text{and}$$

we impose $\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}$.

Step 2. Next we write

$$J(\sigma) = \frac{1}{H}\int_{T}^{T+H} \left(\int_{-\infty}^{\infty} \left| \phi(s) - P^{k}(s) \right|^{2/k} \left| \operatorname{Exp}\left((s-it_0)^{4a+2} \right) \right| dt \right) dt_0$$

where $P(s) = \sum_{\mu_n < X} b_n \mu_n^{-s}$. It is easily seen that $\Delta(X) = O(X^{B})$ and also $1 + O(X^{-B})$ where $B > 0$ is an arbitrary constant and the O-constant depends only on B and a. Again $a_n = \sum_{\mu_{n_1} \cdots \mu_{n_k} = \lambda_n} (b_{n_1} b_{n_2} \cdots b_{n_k})$ and for all $N \geq 1$, we have $\sum_{N \leq \mu_{n_1} \cdots \mu_{n_k} \leq 2N} 1 = O(N^{k})$. From these remarks it is clear that $\phi(s) - P^{k}(s)$ decays fast enough to ensure $J(\sigma) \leq 1$ when σ is large enough. Now from an easy application of a theorem of Gabriel (Theorem 3 above) it follows that in $\sigma \geq \alpha$, $J(\sigma)$ is $\ll_{\varepsilon} [J(\alpha)]^{1-\varepsilon}$ for every positive constant ε uniformly in σ, and so in ($\sigma \geq \beta$, $T \leq t \leq T + H$), $|\phi(s)|$ is bounded above by a constant power of H. (Here for getting the last bound we have to use the fact that for any analytic function $\phi(s)$, $|\phi(s)|^{2/k}$ is bounded by its mean value over a disc of (positive but sufficiently small) constant radius with s as centre).

Step 3. An easy application of a well-known Montgomery–Vaughan theorem (refer [4] for instance) shows that

$$\frac{1}{H}\int_{T}^{T+H} \left(\int_{-\infty}^{\infty} \left| P(s) \right|^{2} \left| \operatorname{Exp}[(s-it_0)^{4a+2}] \right| dt \right) dt_0 = O(V(\sigma) H^{1-2\sigma}).$$

From this and the estimate $J(\sigma) \ll_{\varepsilon} (J(\alpha))^{1-\varepsilon}$ it follows that

$$\int_{\alpha}^{A^3} \left(\frac{1}{H} \int_{T}^{T+H} \left(\int_{-\infty}^{\infty} (|\phi(s) - P^k(s)|^{2/k} + |P(s)|^2) |\mathrm{Exp}((s-it_0)^{4a+2})| \, dt \right) dt_0 \right) d\sigma$$

$$= O((J(\alpha))^{1-\varepsilon} + V(\alpha) H^{1-2\alpha}).$$

Note that $V(\sigma)$ and $V^*(\sigma) H^{1-2\sigma}$ are respectively monotonic increasing and monotonic decreasing functions of σ, where $V^*(\sigma)$ is the same as $V(\sigma)$ with the terms $\mu_n \leqslant 1$ omitted. From now on we assume that $V(\alpha)$ is bounded below by a constant negative power of H. Under this assumption it follows that the integral just considered is

$$O_{\varepsilon}(V(\alpha) H^{1-2\alpha+\varepsilon}) \quad \text{for every positive constant } \varepsilon.$$

Hence there exist intervals I_1 and I_2 contained in $\left(T, T+\frac{H}{10}\right)$ and $\left(T+H-\frac{H}{10}, T+H\right)$ respectively, for which the lengths are $4H^{\varepsilon}$ (δ being any fixed constant satisfying $0 < \delta < \frac{1}{100}$) each, and further $I(I_1, \alpha)$

$$= \int_{\alpha}^{A^3} \left(\frac{1}{H} \int_{I_1} \int_{-\infty}^{\infty} (|\phi(s) - P^k(s)|^{2/k} + |P(s)|^2) |\mathrm{Exp}((s-it_0)^{4a+2})| \, dt \, dt_0 \right) d\sigma$$

and $I(I_2, \alpha)$ defined similarly (by replacing I_1 by I_2) satisfy

$$I(I_1, \alpha) + I(I_2, \alpha) = O(V(\alpha) H^{1-2\alpha+\varepsilon}).$$

Hence by the principle for the mean value over discs referred to in the second step, we see that in ($\beta \leq \sigma \leq A^2$, t in any of the intervals I_1, I_2) we have

$$|\phi(s)|^{2/k} = O(V(\alpha)\, H^{-2\alpha+2\varepsilon+\delta}).$$

Step 4. Let H_1 and H_2 be the mid points of I_1 and I_2 respectively. We now obtain a lower bound for at least one of the mean-values $K(\alpha_1)$ or $K(\alpha_2)$ where $K(\sigma)$ is defined by

$$K(\sigma) = \frac{1}{H_2 - H_1}\int_{H_1}^{H_2} |F_0(s)|^2\, dt, \quad (\beta < \sigma < A^2).$$

Note that when we replace H_1 and H_2 by other points in I_1 and I_2 the mean value $K(\sigma)$ changes by an amount which is at most $O(E)$ where $E = V(\alpha)\, H^{-2\alpha+2\varepsilon+2\delta}$. Hence if σ denotes any of α_2 or α_3 and $H_1 < t < H_2$, we see that if j is a large positive integer constant,

$$(1)\qquad \frac{j!}{2\pi i}\int_{L_0} \frac{F_0(s+w)\,(2X)^w}{w(w+1)\ldots(w+j)}\, dw = \sum_{\mu_n \leq 2X} b_n\, \mu_n^{-s}\left(1 - \frac{\mu_n}{2X}\right)^j,$$

where L_0 is the line Re $w = A^2$. Deform the line L_0 to the contour described by the lines L_1, L_2, L_3, L_4, L_5 (in this order) defined as follows. Let $H_3 = H_1 - H^\delta$, $H_4 = H_2 + H^\delta$ where δ is a small positive constant. L_1 and

L_5 are the portions $\mathrm{Im}\, w \leq -H_3$ and $\mathrm{Im}\, w \geq H_4$ respectively of L_0. L_2 is the line segment

$$(\mathrm{Im}\, w = -H_3,\ \alpha_1 - \sigma \leqslant \mathrm{Re}\, w \leqslant A^2)$$

and L_4 is the line segment

$$(\mathrm{Im}\, w = H_4,\ \alpha_1 - \sigma \leqslant \mathrm{Re}\, w \leqslant A^2).$$

L_3 is the line segment $(\mathrm{Re}\, w = \alpha_1 - \sigma,\ -H_3 \leqslant \mathrm{Im}\, w \leqslant H_4)$. Taking the mean square after deformation of L_0 we find from the equation (1), (Note that the only pole to be taken care of is $w = 0$),

$$(2)\ \begin{cases} K(\sigma) \ll V_1(\sigma)\, H^{1-2\sigma} + (K(\alpha_1) + E)\, H^{-2(\sigma-\alpha_1)} \\ \qquad \text{(where } V_1(\sigma) \text{ is defined below) and also} \\ V(\sigma)\, H^{1-2\sigma} \ll K(\sigma) + (K(\alpha_1) + E)\, H^{-2(\sigma-\alpha_1)}. \end{cases}$$

The reason for this is that the mean square of the RHS of (1) is $\gg$ and $\ll$ $V_1(\sigma)\, H^{1-2\sigma}$ where $V_1(\sigma)$ is defined by

$$V_1(\sigma) = \frac{1}{H} \sum_{\mu_n \leqslant X} |b_n|^2 \left(1 - \frac{\mu_n}{2X}\right)^{2j} \left(\frac{H}{\mu_n}\right)^{2\sigma}.$$

Since $V(\alpha) \leqslant V(\sigma)$ we may omit the term containing E in the second of the equations (2), provided $\alpha_1 - \alpha < \frac{1}{2}$ (which is true because of our assumptions). This gives us

$$V(\sigma)\, H^{1-2\sigma} \ll K(\sigma) + K(\alpha_1)\, H^{-2(\sigma-\alpha_1)};\ (\sigma = \alpha_2, \alpha_3)$$

If we put $\sigma = \alpha_2$ we get a lower bound for one at least of the quantities $K(\alpha_1)$ or $K(\alpha_2)$.

We now deduce from the last inequality

$$V(\sigma)\, H^{1-2\sigma} \ll I(\sigma) + I(\alpha_1)\, H^{-2(\sigma-\alpha_1)},$$

$$(\sigma = \alpha_2, \alpha_3).$$

This is possible since in the range ($\sigma > \alpha_1$, $H_3 < t < H_4$), $|F_0(s)|^2$ is a very good appioximation (in the mean) to $|\phi(s)|^{2/k}$ and we leave the details as an exercise. From the last inequality it follows that $I(\sigma) \gg V(\sigma) H^{1-2\sigma}$, for one at least of the values $\sigma = \alpha_1$ or α_2.

Next by the convexity theorem of Gabriel (Theorem 3), we find that with the value of σ (α_1 or α_2 determined),

$$(3) \quad (I(\sigma))^{\alpha_3-\alpha} \leq (I(\alpha))^{\alpha_3-\sigma} (I(\alpha_3))^{\sigma-\alpha}.$$

Moreover by the arguments used in the first of the inequalities in (2) we get (by taking X in place of 2X in (1)).

$$I(\alpha_3) \ll V(\alpha_3) H^{1-2\alpha_3} + (I(\sigma)+E) H^{-2(\alpha_3-\sigma)} + E.$$

Now $E = V(\alpha) H^{-2\alpha + 2\varepsilon + 2\delta} \leq V(\alpha_3) H^{1-2\alpha_3}$

$\left(\text{since by our assumptions, } \alpha_3 - \alpha < \frac{1}{2}\right)$ by a small choice of the positive constants ε, δ. Thus we get

$$I(\alpha_3) \ll V(\alpha_3) H^{1-2\alpha_3} + I(\sigma) H^{-2(\alpha_3-\sigma)},$$

and so by (3)

$$(4) \quad (I(\sigma))^{\alpha_3-\alpha}$$

$$\ll (I(\alpha))^{\alpha_3-\sigma} \left(V(\alpha_3) H^{1-2\alpha_3} + I(\sigma) H^{-2(\alpha_3-\sigma)}\right)^{\sigma-\alpha}.$$

This holds for either $\sigma = \alpha_1$ or $\sigma = \alpha_2$ and gives us

$$(5)\quad \begin{cases} \text{Either} \quad I(\alpha) \gg V(\sigma)\, H^{1-2\alpha} > V(\alpha)\, H^{1-2\alpha}, \\ \text{or} \quad (I(\alpha))^{\alpha_3-\sigma} \gg (V(\sigma)\, H^{1-2\sigma})^{\alpha_3-\alpha} \\ \qquad\qquad (V(\alpha_3)\, H^{1-2\alpha_3})^{\alpha-\sigma} . \end{cases}$$

The second of these inequalities gives

$$I(\alpha) \gg H^{1-2\alpha}\, V(\sigma) \left\{ \frac{V(\sigma)}{V(\alpha_3)} \right\}^{\frac{\sigma-\alpha}{\alpha_3-\sigma}}$$

Since $V(\alpha) \leqslant V(\alpha_1) \leqslant V(\sigma) \leqslant V(\alpha_3)$ and since $\frac{u-\alpha}{\alpha_3-u}$ is an increasing function of u in $\beta \leqslant u < \alpha_3$, we get finally

$$I(\alpha) \gg V(\alpha)\, H^{1-2\alpha} \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\}^{\frac{\alpha_2-\alpha}{\alpha_3-\alpha_2}}$$

Step 5. Step 4 nearly completes the proof. For we could have started with a slight modification of $I(\sigma)$ by averaging over a slightly smaller interval contained in $(T, T+H)$ instead of $(T, T+H)$. For instance by cutting off bits of length H^{δ} on either side. The decaying factor $\operatorname{Exp}\left((s-it_0)^{4a+2} \right)$ enables us to replace the modified

$$I(\sigma) \text{ by } \frac{1}{H} \int_T^{T+H} |\phi(s)|^{2/k}_{\sigma=\alpha}\, dt$$

in the last lower bound.

Steps 1, 2, 3, 4 and 5 complete the proof of theorem 2.

Theorem 4

With the notation of theorem 2, we have,

$$\frac{1}{H}\int_{T}^{T+H} |\phi(\alpha + it)|^2 \, dt \gg \left(V(\alpha) H^{1-2\alpha} \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\}^{\frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2}} \right)^k$$

where the constant implied by the Vinogradov symbol is effective

Remark. This theorem will be used in [1].

References

1. **R. Balasubramanian And K. Ramachandra,** Some problems of analytic number theory-III, *Hardy-Ramanujan Journal, Vol. 4 (1981).*
2. **D. R. Heath-Brown,** Fractional moments of the Riemann zeta-function, (*to appear*).
3. **K. Ramachandra,** Progress towards a conjecture on the mean-value of Titchmarsh Series, *Proceedings of the Durham Conference on Analytic number theory,* (*July - August 1979*), *(to appear).*
4. **K. Ramachandra** Some remarks on a theorem of Montgomery and Vaughan, *J. of Number theory, Vol. 11 (1975), 465-471.*
5. **E. C. Titchmarsh,** The theory of the Riemann zeta-function, *Oxford (1951).*

Manuscript Completed In The Final Form On 11-Oct-1980

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Colaba, Bombay 400 005
(India).