Progress Towards a conjecture on the mean-value of titchmarsh series-II
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INTRODUCTION

The result of this paper may be considered as complementary to that of my earlier paper [2], on Titchmarsh series. Although not as interesting as the earlier result, the result of the present paper finds a nice application, (See [1]). In [3] I defined a class of series called Titchmarsh series and I now start by recalling its definition.

**Titchmarsh Series.** (or briefly K D T series).

Let $A > 10$ be a constant.

Let $\frac{1}{A} < \mu_1 < \mu_2 < \ldots$ where

$$\frac{1}{A} < \mu_{n+1} - \mu_n < A \quad \text{(for} \quad n = 1, 2, 3, \ldots\text{)}.$$  

(In [2] the notation is slightly different and we have used there $\lambda_n$ instead of $\mu_n$ and for simplicity assumed $\lambda_1 = 1$. Also we have used there $a_1, a_2, \ldots$ in place of our present $b_1, b_2, \ldots$ and assumed for simplicity $a_1 = 1$. We have written there $F(s)$ instead of $F_0(s)$). Let $b_1, b_2, \ldots$ be a sequence of complex numbers possibly depending on a parameter $H > 10$ such that

$$|b_n| < (\mu_n H)^A.$$  

Put $F_0(s) = \sum_{n=1}^{\infty} b_n \mu_n^{-s}$, where $s = \sigma + it$.  


$F_0(s)$ is called a KDT series if there exists a constant $A > 10$ and a system of rectangles $R(T, T + H)$ defined by $\{ \sigma > 0, T < t < T + H \}$ where $10 < H < T$, and $T$ (which may be related to $H$) tends to infinity, and $F_0(s)$ admits an analytic continuation into these rectangles and the maximum of $|F_0(s)|$ taken over $R(T, T + H)$ does not exceed $\text{Exp}(HA)$.

I then made the following conjecture:

**Conjecture**

For a KDT series $F_0(s)$, we have,

$$\frac{1}{H} \int_0^L |F(it)|^2 dt > C_A \sum_{n=1}^\infty |b_n|^2,$$

where $X = 2 + D_A H$, $L$ denotes the side ($\sigma = 0, T < t < T + H$) of $R(T, T + H)$, and $C_A$ and $D_A$ are positive constants depending only on $A$, provided $\mu_1 = b_1 = 1$.

I proved the following theorem.

**Theorem 1**

Under the restrictions $\mu_1 = b_1 = 1$, we have,

$$\frac{1}{H} \int_0^L |F_0(it)|^2 dt > C_A \sum_{n=1}^\infty \frac{|b_n|^2}{n},$$

where $X = 2 + D_A H$ and, $C_A$ and $D_A$ are effective positive constants depending only on $A$.

I now prove the following theorem.
Theorem 2

For some convenience let us assume in the definition of Titchmarsh series $F_0(s)$ the rectangles $R(T, T + H)$ to be $(\sigma > \beta, T < t < T + H)$ where $\beta$ is a positive constant such that $0 < \beta < \alpha_1 < \frac{1}{2}$, where $\alpha_1$ is another constant. Let $k \geq 2$ be an integer and write $F(s) = (F_0(s))^k = \sum_{n=1}^{\infty} (a_n \lambda_n^{-s})$, a series which is surely convergent where $F_0(s)$ is absolutely convergent. Put $Y = (M + H)$ where

$$\lambda = kA \quad (\alpha_1 - \beta)$$

and $M = \text{maximum of } |F_0(s)|$ taken over $R(T, T + H)$. Define the entire function $\phi(s)$ by

$$\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta \left( \frac{Y}{\lambda_n} \right)$$

where for $X > 0$, $\Delta(X)$ is defined by

$$\Delta(X) = \frac{1}{2\pi i} \int_{2 - i \infty}^{2 + i \infty} X^W \text{Exp} \left( W^{4a+2} \right) \frac{dW}{W} ,$$

$a$ being a suitable positive integer constant at our choice. We now suppose that $\alpha, \beta, \alpha_1, \alpha_2, \alpha_3$ are constants satisfying

$$\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2} .$$

Put $X = [2^A - 100]^H + 2$ and

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n|^2 \left( \frac{H}{\mu_n} \right)^{2\sigma}.$$
Then we have,

\[
\frac{1}{H} \int_1 \phi (\xi + it) \left| \frac{2k}{1 - \phi^2} \right| dt
\]

\[
\gg V(\xi) H^{1-2\lambda} \left\{ \frac{V(\xi_1)}{V(\xi_3)} \right\}^{d_2 - d_1}
\]

\[
\gg \frac{d_2 - d_1}{d_3 - d_2}
\]

provided only that \( V(\xi) \neq 0 \) and \( (\log V(\xi)) (\log H)^{-1} \) is bounded below by a negative constant. Here the constant implied by the Vinogradov symbol \( \gg \) depends only on \( A, k, \xi, \beta, \xi_1, \xi_2, \xi_3 \) and the negative constant referred to just now. Further it is effective.

Remark: It will be clear from the proof that if

\( \mu_n = n (n = 1, 2, 3, \ldots) \) then we can choose \( X = \left\lfloor \frac{H}{100} \right\rfloor + 2. \)

The object of this paper is to prove theorem 2. The proof of the theorem is fairly long. The proof depends upon a special case of a convexity theorem of R. M. Gabriel which we state below (in the notation of D. R. Heath-Brown's paper [2]; for the more general theorem of Gabriel see the reference in [2] or Titchmarsh's famous book [5] pages 203 and 337.)

**Theorem 3**

Let \( f(z) \) be regular in the infinite strip \( \lambda < \text{Re} \ z < \beta \) and continuous for \( \lambda < \text{Re} \ z < \beta \). Suppose \( f(z) \to 0 \) as \( |\text{Im} \ z| \to \infty \), uniformly in \( \lambda < \text{Re} \ z < \beta \). Then for \( \lambda < \gamma < \beta \) and any \( q > 0 \), we have
\[
\int_{-\infty}^{\infty} \left| f(\gamma + it) \right|^q dt < \left( \int_{-\infty}^{\infty} \left| f(\alpha + it) \right|^q dt \right)^{\frac{B-\gamma}{B-\alpha}} \left( \int_{-\infty}^{\infty} \left| f(\beta + it) \right|^q dt \right)^{\frac{\gamma-\alpha}{\beta-\alpha}},
\]

provided the right hand side is finite.

Apart from this we have to use a well-known theorem of Montgomery and Vaughan. For reference see for instance my paper [4], where I give a simple proof of a weaker result which is sufficient for the purposes of this paper.

We now split up the proof of theorem 2 into several steps and give a brief sketch of these steps.

§ 2. Proof of Theorem 2

Step 1. Let

\[
I(\sigma) = \frac{1}{H} \int_{T+H}^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) \right|^2/k \right) dt_0,
\]

and assume that \( I(\alpha) < \mathcal{V}(\alpha) H^{1-2\alpha} \). The constant \( \alpha \) shall be a sufficiently large positive integer. As already stated we set

\[
X = \left[ 2^{-A-100} H \right] + 2, \text{ and }
\]

\[
V(\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} \left| b_n \right|^2 \left( \frac{H}{\mu_n} \right)^{2\sigma}
\]

and

we impose \( \alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2} \).
Step 2. Next we write

\[ J(\sigma) = \frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) - P_k(s) \right|^2 \exp\left( (s-i\tau_{0})^{4a+2} \right) \, dt \right) \, dt_0 \]

where \( P(s) = \sum b_n \mu_n^{-s} \). It is easily seen that \( \Delta(X) = O(X^B) \) and also \( 1 + O(X^{-B}) \) where \( B > 0 \) is an arbitrary constant and the \( O \)-constant depends only on \( B \) and \( a \).

Again \( a_n = \mu_{n_1} \cdots \mu_{n_k} = \lambda_n (b_{n_1} b_{n_2} \cdots b_{n_k}) \) and for all \( N > 1 \), we have \( \sum_{N < \mu_{n_1} \cdots \mu_{n_k} \leq 2N} 1 = O(N^k) \). From these remarks it is clear that \( \phi(s) - P_k(s) \) decays fast enough to ensure \( J(\sigma) < 1 \) when \( \sigma \) is large enough. Now from an easy application of a theorem of Gabriel (Theorem 3 above) it follows that in \( \sigma > \alpha \), \( J(\sigma) \) is \( \ll_\varepsilon [J(\alpha)]^{1-\varepsilon} \) for every positive constant \( \varepsilon \) uniformly in \( \sigma \), and so in \( (\sigma > \beta, T < t < T + H) \), \( |\phi(s)| \) is bounded above by a constant power of \( H \). (Here for getting the last bound we have to use the fact that for any analytic function \( \phi(s) \), \( |\phi(s)|^{2/k} \) is bounded by its mean value over a disc of (positive but sufficiently small) constant radius with \( s \) as centre).

Step 3. An easy application of a well-known Montgomery-Vaughan theorem (refer [4] for instance) shows that

\[ \frac{1}{H} \int_{T}^{T+H} \left( \int_{-\infty}^{\infty} |P(s)|^2 \exp\left( (s-i\tau_{0})^{4a+2} \right) \, dt \right) \, dt_0 = O(V(\sigma) H^{1-2\sigma}). \]
From this and the estimate $J(\sigma) \ll \varepsilon (J(\sigma))^{1-\varepsilon}$ it follows that

$$
A^s \int_{\mathcal{A}} \left( \frac{1}{H} \int_{T}^{T+H} \int_{-\infty}^{\infty} (|\phi(s) - P(s)|^{2/k} + |P(s)|^{2} \left| \text{Exp} \left( (s - it_0)^{4a+2} \right) \right| dt_0 \right) d\sigma
$$

$$
\quad = O ((J(\mathcal{A}))^{1-\varepsilon} + V(\mathcal{A}) H^{1-2A}).
$$

Note that $V(\sigma)$ and $V^*(\sigma) H^{1-2\sigma}$ are respectively monotonic increasing and monotonic decreasing functions of $\sigma$, where $V^*(\sigma)$ is the same as $V(\sigma)$ with the terms $P_n \ll 1$ omitted.

From now on we assume that $V(\mathcal{A})$ is bounded below by a constant negative power of $H$. Under this assumption it follows that the integral just considered is

$$O_\varepsilon (V(\mathcal{A}) H^{1-2A + \varepsilon}) \text{ for every positive constant } \varepsilon.$$

Hence there exist intervals $I_1$ and $I_2$ contained in

$$\left(T, T + \frac{H}{10}\right) \text{ and } \left(T + H - \frac{H}{10}, T + H\right)$$

respectively, for which the lengths are $4H^\varepsilon$ ($\delta$ being any fixed constant satisfying $0 < \delta < \frac{1}{100}$) each, and further $I(I_1, \mathcal{A})$

$$= \int_{\mathcal{A}} \left( \frac{1}{H} \int_{I_1}^{I_1 + H} \int_{-\infty}^{\infty} (|\phi(s) - P(s)|^{2/k} + |P(s)|^{2} \left| \text{Exp} \left( (s - it_0)^{4a+2} \right) \right| dt_0 \right) d\sigma$$

and $l(I_2, \mathcal{A})$ defined similarly (by replacing $I_1$ by $I_2$) satisfy

$$l(I_1, \mathcal{A}) + l(I_2, \mathcal{A}) = O(V(\mathcal{A}) H^{1-2A + \varepsilon}).$$
Hence by the principle for the mean value over discs referred to in the second step, we see that in \((\beta \leq \sigma \leq A^2,\ t \text{ in any of the intervals } I_1, I_2)\) we have

\[
|\phi(s)|^{2/k} = O\left( V(a) H^{-2a} + 2E + \delta \right).
\]

**Step 4.** Let \(H_1\) and \(H_2\) be the mid points of \(I_1\) and \(I_2\) respectively. We now obtain a lower bound for at least one of the mean-values \(K(a_1)\) or \(K(a_2)\) where \(K(\sigma)\) is defined by

\[
K(\sigma) = \frac{1}{H_2 - H_1} \int_{H_1}^{H_2} |F_0(s)|^2 \, ds, \quad (\beta < \sigma \leq A^2).
\]

Note that when we replace \(H_1\) and \(H_2\) by other points in \(I_1\) and \(I_2\) the mean value \(K(\sigma)\) changes by an amount which is at most \(O(E)\) where \(E = V(a) H^{-2a} + 2E + 2\delta\). Hence if \(\sigma\) denotes any of \(a_2\) or \(a_3\) and \(H_1 < t < H_2\), we see that if \(j\) is a large positive integer constant,

\[
(1) \quad \frac{j!}{2\pi i} \int_{L_0} \frac{F_0(s + w)(2X)^w}{w(w + 1) \ldots (w + j)} \, dw = \sum_{\mu_n \leq 2X} b_n \mu_n^{-s} \left( 1 - \frac{\mu_n}{2X} \right)^j,
\]

where \(L_0\) is the line \(\text{Re } w = A^2\). Deform the line \(L_0\) to the contour described by the lines \(L_1, L_2, L_3, L_4, L_5\) (in this order) defined as follows. Let \(H_3 = H_1 - \delta\), \(H_4 = H_2 + \delta\) where \(\delta\) is a small positive constant. \(L_1\) and
L_5 are the portions \( \text{Im } w \leq -H_3 \) and \( \text{Im } w \geq H_4 \) respectively of \( L_0 \). \( L_2 \) is the line segment
\[
( \text{Im } w = -H_3, \quad \sigma_1 - \sigma \leq \text{Re } w \leq A^2 )
\]
and \( L_4 \) is the line segment
\[
( \text{Im } w = H_4, \quad \sigma_1 - \sigma \leq \text{Re } w \leq A^2 ).
\]
\( L_3 \) is the line segment (\( \text{Re } w = \sigma_1 - \sigma, \quad -H_3 \leq \text{Im } w \leq H_4 \)).

Taking the mean square after deformation of \( L_0 \) we find from the equation (1), (Note that the only pole to be taken care of is \( w = 0 \)),

\[
\begin{align*}
K ( \sigma ) & \ll V_1 ( \sigma ) H^{1 - 2 \sigma} + (K (\sigma_1) + E) H^{-2(\sigma - \sigma_1)} \\
\text{(where } V_1 ( \sigma ) \text{ is defined below) and also} \\
V ( \sigma ) H^{1 - 2 \sigma} & \ll K (\sigma) + (K (\sigma_1) + E) H^{-2(\sigma - \sigma_1)}.
\end{align*}
\]

The reason for this is that the mean square of the RHS of (1) is \( \gg \) and \( \ll V_1 ( \sigma ) H^{1 - 2 \sigma} \) where \( V_1 ( \sigma ) \) is defined by

\[
V_1 ( \sigma ) = \frac{1}{H} \sum_{\mu_n \leq X} |b_n|^2 \left( - \frac{\mu_n}{2X} \right)^2 n \frac{H^2}{(\mu_n)^2}.
\]

Since \( V (\sigma_1) \ll V (\sigma) \) we may omit the term containing \( E \) in the second of the equations (2), provided \( \sigma_1 - \sigma < \frac{1}{1} \) (which is true because of our assumptions). This gives us

\[
V ( \sigma ) H^{1 - 2 \sigma} \ll K (\sigma) + K (\sigma_1) H^{-2(\sigma - \sigma_1)}; \quad (\sigma = \sigma_2, \sigma_3)
\]

If we put \( \sigma = \sigma_2 \) we get a lower bound for one at least of the quantities \( K (\sigma_1) \) or \( K (\sigma_2) \).

We now deduce from the last inequality

\[
V ( \sigma ) H^{1 - 2 \sigma} \ll I (\sigma) + I (\sigma_1) H^{-2(\sigma - \sigma_1)},
\]

\( (\sigma = \sigma_2, \sigma_3) \).
This is possible since in the range \((\sigma > \lambda_1, \ H_3 < t < H_4)\),
\(|F_0(s)|^2\) is a very good approximation (in the mean) to
\(|\phi(s)|^{2/k}\) and we leave the details as an exercise. From the
last inequality it follows that \(I(\sigma) \gg V(\sigma) H^{1-2\sigma}\), for one
at least of the values \(\sigma = \lambda_1\) or \(\lambda_2\).

Next by the convexity theorem of Gabriel (Theorem 3),
we find that with the value of \(\sigma (\lambda_1\) or \(\lambda_2\) determined),

\[
(3) \quad (I(\sigma))^{\lambda_3 - \lambda} < (I(\lambda) - \sigma) (I(\lambda_3))^{\sigma - \lambda}.
\]

Moreover by the arguments used in the first of the
inequalities in (2) we get (by taking \(X\) in place of \(2X\) in (1)).

\[
I(\lambda_3) \ll V(\lambda_3) H^{1-2\lambda_3} + (I(\sigma) + E) H^{-2(\lambda_3 - \sigma)} + E.
\]

Now \(E = V(\lambda) H^{1-2\lambda_3} + E < V(\lambda_3) H^{1-2\lambda_3}\)
(since by our assumptions, \(\lambda_3 - \lambda < \frac{1}{2}\)) by a small choice
of the positive constants \(\epsilon, \delta\). Thus we get

\[
I(\lambda_3) \ll V(\lambda_3) H^{1-2\lambda_3} + I(\sigma) H^{-2(\lambda_3 - \sigma)},
\]

and so by (3)

\[
(4) \quad (I(\sigma))^{\lambda_3 - \lambda} < (I(\lambda))^{\lambda_3 - \sigma} (V(\lambda_3) H^{1-2\lambda_3} + I(\sigma) H^{-2(\lambda_3 - \sigma)})^{\sigma - \lambda}.
\]

This holds for either \(\sigma = \lambda_1\) or \(\sigma = \lambda_2\) and gives us
Either \( I(\sigma) \gg V(\sigma) H^{1-2\sigma} > V(\sigma) H^{1-2\beta} \),
\[
\text{or } (I(\sigma)) \gg (V(\sigma) H^{1-2\beta} - \sigma) (V(\sigma) H^{1-2\beta} - \sigma).
\]

The second of these inequalities gives
\[
\frac{\sigma - \beta}{\beta - \sigma} I(\sigma) \gg H^{1-2\beta} V(\sigma) \left\{ \frac{V(\sigma)}{V(\sigma_3)} \right\}
\]

Since \( V(\sigma) < V(\sigma_1) < V(\sigma) < V(\sigma_3) \) and since \( \frac{u - \beta}{\beta - \sigma} \) is an increasing function of \( u \) in \( \beta < u < \beta_3 \), we get finally
\[
\frac{\beta_2 - \beta}{\beta_3 - \beta_2} I(\sigma) \gg V(\sigma) H^{1-2\beta} \left\{ \frac{V(\sigma_1)}{V(\sigma_3)} \right\}
\]

**Step 5.** Step 4 nearly completes the proof. For we could have started with a slight modification of \( I(\sigma) \) by averaging over a slightly smaller interval contained in \( (T, T + H) \) instead of \( (T, T + H) \). For instance by cutting off bits of length \( H^8 \) on either side. The decaying factor \( \text{Exp} \left( (s - it_0)^{4a + 2} \right) \) enables us to replace the modified
\[
I(\sigma) \text{ by } \frac{1}{H} \int_T^{T+H} |\phi(s)| \sigma^{2/k} \, dt
\]
in the last lower bound.

Steps 1, 2, 3, 4 and 5 complete the proof of theorem 2.
Theorem 4

With the notation of theorem 2, we have,

\[
\frac{1}{H} \int_{T}^{T+H} \left| \phi \left( \alpha + it \right) \right|^2 dt \gg \\
\left( V(\alpha) H^1 - 2\alpha \left( \frac{V(\alpha_1)}{V(\alpha_3)} \right) \right)^k \frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_1}
\]

where the constant implied by the Vinogradov symbol is effective.

Remark. This theorem will be used in [1].

References


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