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ON A QUESTION OF RAMACHANDRA

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Let $a_k(n)$ be the Dirichlet series coefficients defined by the relation

$$(1) \qquad \left(\sum_p \frac{1}{p^s}\right)^k = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} \qquad (\mathrm{Re}\, s > 1).$$

Here k is a positive integer, and we see more explicitly that if $n = p_1^{a_1} \cdots p_t^{a_t}$ then

$$(2) \qquad a_k(n) = \begin{cases} \dfrac{k!}{a_1! \ldots a_t!} & \text{if } \Omega(n) = k, \\ 0 & \text{otherwise,} \end{cases}$$

With possible applications to the Riemann zeta function in mind, Ramachandra has wanted to know the asymptotic size of the quantity

$$\max_k \left(\sum_{n<N} \frac{a_k(n)^2}{n^{2\sigma}}\right)^{\frac{1}{2k}}$$

as a function of N and σ, where σ is fixed, $\frac{1}{2} < \sigma < 1$. We settle this question by demonstrating the following

Theorem: Let σ be fixed, $\frac{1}{2} < \sigma < 1$. With the $a_k(n)$ defined by (1), we have

$$(3) \qquad \max_k \left(\sum_{n<N} \frac{a_k(n)^2}{n^{2\sigma}}\right)^{\frac{1}{2k}} \asymp \frac{(\log N)^{1-\sigma}}{\log\log N}$$

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In addition, for any integer $k > 1$

$$(4) \qquad \Big(\sum_n \frac{a_k(n)^2}{n^{2\sigma}} \Big)^{\frac{1}{2k}} \asymp \frac{k^{1-\sigma}}{(\log k)^{\sigma}}$$

Here the implicit constants may depend only on σ.

With a little more care one could show that the values of k for which the maximum in (3) is attained satisfy

$$k \asymp (\log N)/\log\log N.$$

To establish that the right hand side majorizes the left above we shall require the following two lemmas.

Lemma 1 : Let $Y > 2$, and put

$$R = R(Y) = \{ r : p \mid r \Rightarrow p < Y \}.$$

Then for any $k > 1$, and fixed σ, $\frac{1}{2} < \sigma < 1$,

$$\Big(\sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \Big)^{\frac{1}{2k}} \ll \frac{Y^{1-\sigma}}{\log Y}.$$

Proof: Let $X_2, X_3, \dots, X_p$ be independent random variables, each uniformly distributed on the circle $|z| = 1$. Then

$$\Big(\sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \Big)^{\frac{1}{2k}} = \| \sum_{p<y} \frac{X_p}{p^{\sigma}} \|_{2k} < \| \sum_{p<y} \frac{X_p}{p^{\sigma}} \|_{\infty}$$

$$= \sum_{p<y} \frac{1}{p^{\sigma}} \ll \frac{Y^{1-\sigma}}{\log Y}.$$

Lemma 2 : Let $Y > 2$, and put

$$S = S(y) = \{ s : p \mid s \Rightarrow p > y \}.$$

Then for any $k > 1$, and fixed σ, $\frac{1}{2} < \sigma < 1$,

$$\left(\sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2k}} \ll \frac{k^{1/2} Y^{\frac{1}{2} - \sigma}}{(\log Y)^{1/2}}$$

Proof : From (2) we see that $a_k(s) < k!$ for all s.

Thus

$$\sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} < k! \sum_{s \in S} \frac{a_k(s)}{s^{2\sigma}} = k! \left(\sum_{p > Y} \frac{1}{p^{2\sigma}} \right)^k$$

$$< k! \left(\frac{c Y^{1 - 2\sigma}}{\log Y} \right)^k.$$

and the stated bound follows on taking the 2k-th root.

We now prove the Theorem. We first show that the left hand side of (3) is at least as large as the right hand side. Let Y be the largest integer such that

$$P = \prod_{p \leq Y} p \leq N.$$

Then by the prime number theorem with remainder,

$$Y = \log N + O\left(\frac{\log N}{(\log \log N)^A} \right)$$

Take $k = \Omega(P) = \pi(Y)$. Then

$$\sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} > \frac{a_k(P)^2}{P^{2\sigma}} = \frac{k!^2}{P^{2\sigma}} > \frac{k!^2}{N^{2\sigma}}$$

so that $$\left(\sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \gg \frac{k}{N^{\sigma/k}}$$

Clearly $k \sim \dfrac{\log N}{\log\log N}$, and by more careful use of the prime number theorem we see that $k > \dfrac{\log N}{\log \log N}$ for all large N. Thus $N^{1/k} \leq \log N$, and we have the desired lower bound. In (4) the value of k is prescribed ; we choose Y so that $\pi(Y) = k$, we take $N = P$, and proceed as above. We obtain the desired lower bound since

$$\frac{(\log N)^{1-\sigma}}{\log\log N} \sim \frac{k^{1-\sigma}}{(\log k)^{\sigma}}$$

We now complete the proof of (3). Let R and S be as in Lemmas 1 and 2, where Y is a parameter to be chosen later. Any n is uniquely of the form $n = rs$ with $r \in R$, $s \in S$. If $\Omega(n) = k$ and $\Omega(r) = m$ then by (2)

$$a_k(n) = \binom{k}{m} a_m(r)\, a_{k-m}(s).$$

Hence

$$\sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} = \sum_{m=0}^{k} \binom{k}{m}^2 \sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \sum_{\substack{s \in S \\ rs \leq N}} \frac{a_{k-m}(s)^2}{s^{2\sigma}}$$

Since $\binom{k}{m} < 2^k$, this is

$$< 2^k \sum_{m=0}^{k} \binom{k}{m} \left(\sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \right) \left(\sum_{\substack{s \in S \\ s \leq N}} \frac{a_{k-m}(s)^2}{s^{2\sigma}} \right)$$

$$(5) \qquad < 2^k (R^2 + s^2)^k ,$$

where

$$R = \max_m \left(\sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2m}}$$

and

$$S = \max_m \left(\sum_{\substack{s \in S \\ s < N}} \frac{a_m(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2m}}$$

By Lemma 1

$$(6) \qquad R \ll \frac{Y^{1-\sigma}}{\log Y}$$

To treat S we note that $\Omega(s) = m$, $s \in S$, then $s > Y^m$. If $m > \frac{\log N}{\log Y}$ then it follows that $s > N$, and the sum defining S will be empty. Thus we may suppose that $m < \frac{\log N}{\log Y}$. From this bound and Lemma 2 we see that

$$(7) \qquad S \ll \frac{(\log N)^{\frac{1}{2}} Y^{\frac{1}{2} - \sigma}}{\log Y}$$

On combining (5), (6), and (7) it follows that

$$\left(\sum_{n<N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \ll \frac{Y^{1-\sigma}}{\log Y} + \frac{(\log N)^{\frac{1}{2}} Y^{\frac{1}{2} - \sigma}}{\log Y},$$

and it suffices to take $Y = \log N$.

To complete the proof of (4) we may proceed more simply. With the X_p as in the proof of Lemma 1 we see that

$$\left(\sum_{n} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} = \left\| \sum_{p} \frac{X_p}{p^{\sigma}} \right\|_{2k}.$$

By the triangle inequality this is

$$< \Big\| \sum_{p<Y} \frac{X_p}{p^{\sigma}} \Big\|_{2k} + \Big\| \sum_{p>Y} \frac{X_p}{p^{\sigma}} \Big\|_{2k}$$

$$= \left(\sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2k}} + \left(\sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2k}}$$

By Lemmas 1 and 2 this is

$$\ll \frac{Y^{1-\sigma}}{\log Y} + \frac{k^{\frac{1}{2}} Y^{\frac{1}{2}-\sigma}}{(\log Y)^{\frac{1}{2}}}$$

and it suffices to take $Y = k \log k$.

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