Mean-value of the Riemann zeta-function and other remarks III
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§ 1. Introduction

Regarding the mean value lower bounds for
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k}, \]
Titchmarsh was the first to prove (see Theorem 29 on p. 42 of [7]) that
\[ \frac{1}{T} \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} \, dt > C_k (\log T)^k, \quad (k > 0), \quad (1) \]

where \( T > 100 \) and \( k \) is positive integer and \( C_k > 0 \) depends only on \( k \). (However, Theorem 29 is stated without proof in [6], and the reference in [6] to published papers referring to Theorem 29 indicate that he proved \( \lim \sup \frac{\text{LHS}}{\text{RHS}} > 0 \) as \( T \to \infty \).)

In many of my recent papers ([2, I, II], [3, I, II]) I considered extension of (1) to non-integral values of \( k \) and more general questions. In particular I proved that (1) holds for all \( k > 0 \) for which \( 2k \) is an integer. (I proved also that (1) holds for all \( k > 0 \) on Riemann hypothesis.) My proof of this was self-contained and did not make use of Gabriel’s convexity theorem. Heath-Brown noticed [1] that the use of Gabriel’s theorem (for reference to Gabriel’s theorem see [I]) would not only simplify my proof of (1) for integral \( k \) and half an odd integer, but would yield further dividends like the proof of (1) for all rational \( k \). In my papers [2, I]
and [3, II], I dealt with the case $k$ irrational and proved for the LHS of (1) the lower bound $C_k \left( \frac{\log T}{\log \log T} \right)^{k^2}$. It is the purpose of this note to give a proof of a more illuminating lower bound $C_k \left( \frac{\log T}{q_n} \right)^{k^2}$ (to be made precise) which is slightly a better lower bound. It is still very far from the lower bound $C_k (\log T)^{k^2}$. The present lower bound is made possible by Heath-Brown's idea [1] of using Gabriel's theorem to prove (1) for all rational $k$. As in our earlier papers we start with the fundamental function

$$M(\alpha) = M(\alpha, k, T, T+H) = \max_{\sigma > \alpha} \left( \frac{1}{H} \int_{T}^{T+H} \zeta(\sigma + it) |t|^{2k} dt \right),$$

(2)

where $T > H > 0$, $\alpha > \frac{1}{2}$ and $k > 0$. We prove the following (Hereafter we suppose $k > 0$ and $k$ is irrational) theorem.

**Theorem 1:**

Let $H_0 = H + 10000$ Then

$$M\left( \frac{1}{2} \right) > C_k \left( \frac{\log H_0}{q_n} \right)^{k^2}$$

(3)

where $C_k > 0$ depends only on $k$. Here $\frac{p_m}{q_m}$ is the $m^{th}$ convergent to the simple continued fraction expansion of $k$ and $n$ is the unique integer such that $q_n q_{n+1} > \log \log H_0 > q_{n-1} q_n$. 
From this it is possible to deduce (as a very general principle applicable in many situations) with the help of Gabriel's theorem (using a suitable kernel function) the following theorem (See § 2)

**Theorem 2**: If \( T > H > 10000 \) \( \log \log T \geq 10^8 \), then we have,

\[
\frac{1}{T} \int_{H}^{T+H} | \zeta \left( \frac{1}{2} + it \right) |^{2k} \, dt > C_k \left( \frac{\log H}{q_n} \right)^k,
\]

where \( C_k \) and \( q_n \) are as in theorem 1.

**Remark 1**: Since \( \log \log H_0 > q_n q_n-1 \) we can replace the RHS in (4) by \( C_k \left( \frac{q_{n-1}}{\log \log H_0} \right)^k \). However \( q_n - 1 \) cannot in general be replaced by a more explicit function of \( H_0 \).

**Remark 2**: If the s.c.f. expansion of \( k \) has bounded partial quotients, then it is well-known that \( q_n \) is around \( (\log \log H_0)^{\frac{1}{2}} \) and hence we can replace the RHS in (4) by \( C_k \left( \log H_0 \right)^{k^2} (\log \log H_0)^{-\frac{1}{2} k^2} \) for such \( k \). By an extension of this argument we note that (by using a deep theorem of K. F. Roth [6]) that we can replace the RHS of (4) by \( C_k \left( \log H_0 \right)^{k^2} (\log \log H_0)^{-\frac{1}{2} k^2 - \varepsilon} \), where \( \varepsilon > 0 \), \( C_k, \varepsilon > 0 \) provided \( k \) is algebraic Roth's result however readers \( C_k \) to be an ineffective constant for all small constants \( \varepsilon > 0 \).
Remark 3. Theorems 1 and 2 have their extensions to $L$-series and hybrid analogues and we do not bother to state them here.

Remark 4. As remarked already Theorem 2 shows that

$$\lim_{T \to \infty} \left( \frac{\psi(T)}{T \left( \log T \right)^k} \frac{2T}{\zeta(\frac{1}{2} + it)} \int \frac{2k \, dt}{T} \right)$$

is infinity if

$$\psi(T) = \left( \log \log T \right)^{\frac{k}{2}}.$$  

Theorem 2 also shows that if

$$\psi(T) = \left( \log \log T \right)^{\frac{k-2}{2}}$$

then $\limsup (\ldots) > 0$. Further if $T \to \infty$

$\psi(T)$ is any function which tends to infinity as $T$ then there exists an irrational $k$ (depending on the nature of the function $\psi$) such that $\limsup (\ldots) = \infty$.

Remark 5: For any natural number $r$ in its lowest terms, let $H(r)$ denote the sum of the absolute values of the numerator and the denominator and 100. Write $L_1(x) = \log x$, $L_2(x) = \log \log x$ and so on. Consider the intervals

$$[r - \delta(r), r + \delta(r)]$$

where $\delta(r) = \varepsilon (H(r))^{-2} (L_1(H(r)))^{-1} (L_2(H(r)))^{-2}$ where $\varepsilon$ is any constant satisfying $0 < \varepsilon < \frac{1}{100}$. The sum of the lengths of these intervals is $\leq 100 \varepsilon$ (as can be easily seen by taking all rationals $r$ with $H(r)$ lying in \( \left[ 2^m, 2^m + 1 \right] \), $m = 1, 2, 3, \ldots$). This observation is due to Khintchine and gives the following corollary to Theorems 1 and 2. Namely we can replace the RHS in (3) and (4) by

$$C_k \left( L_1(H_0)(L_4(H_0))^{-1} \right)^k (L_2(H_0)L_3(H_0))^{-\frac{k}{2}}$$
for almost all $k$. This corollary was pointed out to me by Dr. M. Ram Murthy.

For some more results see § 4. These deal with Titchmarsh series.

§ 2. Deduction of Theorem 2 From Theorem 1

Theorem 1 is more fundamental. Depending on the availability of bounds for and suitable kernels we can deduce Theorem 2 from Theorem 1, with the help of Gabriel's Theorem viz

**Theorem 3**

(R. M. Gabriel) Let $f(z)$ be regular in the infinite strip $\alpha < \text{Re} \, z < \beta$ and continuous in $\alpha < \text{Re} \, z < \beta$. Suppose $|f(z)| \to 0$ uniformly as $z \to \infty$ in $\alpha < \text{Re} \, z < \beta$. Then for any $q^* > 0$, we have

\[
\int_0^\infty |f(\gamma + it)|^{q^*} \, dt < \left( \int_\alpha^\beta |f(\alpha + it)|^{q^*} \, dt \right)^\lambda \left( \int_\infty^\infty |f(\beta + it)|^{q^*} \, dt \right)^\mu,
\]

where $\lambda = \frac{\beta - \gamma}{\beta - \alpha}$ and $\mu = \frac{\gamma - \alpha}{\beta - \alpha}$

(Notice that the integrals are not required to converge. If the RHS is finite then $s_0$ is LHS).

Later in § 4 we state Theorem 3' (also due to Gabriel) from which he deduced Theorem 3. Both the theorems 3 and 3' were used by Heath-Brown in [1]

**Remark:** If $\phi(w)$ is a suitable kernel function (which is analytic) then replacing $|f'(z)|^{q^*}$ by $|f(z) \phi(w - z)|^{q^*}$ we can replace the three integrals $|f(\alpha + it) \phi(\alpha - x - it)|^{q^*}$ with $x = \gamma$, $\alpha$ and $\beta$ respectively. This would mean (in a
certain sense) a localisation in a short neighbourhood of ψ (the shortest depending on the fastness of tapering of the function ϕ (w)). Hence if we now average both sides of the inequality with respect to ψ over a short interval we get a convexity result for the mean-value of |f (x+it)|^q (x=γ, λ and β) as x varies over a short interval. Thus we state the following theorem.

Theorem 4:

We have, by taking f (z) = ζ (z) = \frac{1}{z-1} and ϕ (w) =

\[ \exp \left\{ \left( \sin \left( \frac{w}{100} \right) \right)^2 \right\}, \quad \lambda = \frac{1}{2}, \quad \beta = 2 \text{ and } \gamma \text{ any}
\]

number in [λ, β], we have,

\[ \frac{1}{H} \int_{T}^{T+H} |ζ (γ + it)|^q \, dt \ll
\]

\[ \left( \frac{1}{H} \int |ζ (λ + it)|^q \, dt \right)^\lambda \left( \frac{1}{H} \int |ζ (β + it)|^q \, dt \right)^\mu
\]

where T > H > 10000 \log \log T > 10^8, and the constant implied by \ll depends only on q*.

From Theorem 4 we can easily deduce Theorem 2 as a corollary to Theorem 1.

§ 3 Proof of Theorem 1. We give a brief sketch. We can assume without loss of generality that H exceeds a sufficiently large constant since |ζ (10 + it) | > \frac{1}{2}. We can now write H for H_0.

Lemma 1 \[ \ln \left( \sigma > \frac{1}{2} + \frac{1}{\log H} \right), \quad T + 1 < t < T + H - 1 \]
we divide the t-range into intervals I of length \((\log H)^A\) (where \(A > 0\) is a suitable constant) ignoring a bit at one end if necessary. Denote the maximum of \(| \zeta(s) |\) in \(\sigma > \frac{1}{2} + \frac{1}{\log H}\), \(t \in I\) by \(\mu(I)\). Then

\[
\sum_{I} \mu(I) \cdot 2^k \leq H \cdot M \left( \frac{1}{2} \right) \cdot (\log H)^2.
\]

**Remark:** Hereafter we assume that \(M \left( \frac{1}{2} \right) < (\log H)^k\) since otherwise there is nothing to prove. Hence the LHS is

\[
\leq (\log H)^k^2 + 2
\]

**Proof:** Follows from the fact that \(| \zeta(s) |\) does not exceed its mean-value over a disc with centre \(s\) and radius \(\frac{1}{2} + 2 - 2Bk\).

**Lemma 2:** Let \(B > 0\) be a constant. Then the number of intervals \(I\) for which \(\mu(I) > (\log H)^B\), is not more than \(2^2 + 2 - 2Bk\). Denote any of the remaining intervals \(J\). Then

\[
\sum_{J} \left( \int_{J} | \zeta(s) |^{2k} \, dt \right) \geq \sum_{J} \left( \int_{J} \left| \zeta(s) \right|^{2a} \, dt + O(\log H)^A \right),
\]

where \(p = p_n\), \(q = q_n\), \(a = \frac{p}{q}\) and \(s = \frac{1}{2} + \frac{q}{\log H} + it\).

**Proof:** The first part follows from Lemma 1. The second part follows from the remark

\[
\int_{J} \left| \zeta(s) \right|^{2k} \, dt \geq \int_{J} \left| \zeta(s) \right|^{2a} \, dt
\]

\[
> \int_{J} \left| \zeta(s) \right|^{2a} - (\log H)^A
\]

which is valid since, if \((\log H)^B > | \zeta(s) | > 1\), then

\[
| \zeta(s) |^{2k - 2a} = \text{Exp} \left( (2k - 2a) \log | \zeta(s) | \right) \geq \text{Exp} (-2B).
\]
(We have to use the well-known result
\[ |k - a| < \frac{1}{q_n q_{n+1}} \left( < \frac{1}{\log \log H} \right) \]
from the theory of continued fractions).

**Lemma 3** We have, with \( s = \frac{1}{2} + \frac{q}{\log H} + it \),
\[ \sum_{j} \int_{j} \frac{2k}{\zeta(s)} \, dt \gg \sum_{j} \int_{j} \frac{2a}{\zeta(s)} \, dt + O(H) \]
where \( a = \frac{p_n}{q_n} \), and \( q = q_n \). The same inequality is also valid if \( s = \frac{1}{2} + \frac{Dq}{\log H} + it \) where \( D > 0 \) is a constant.

**Proof:** Follows from [Lemma 2](#).

The rest of the work consists in proving that if \( D \) is large enough the RHS in the inequality of Lemma 3 is
\[ \gg H (\log H) k^2 q^{-k^2} \]
for at least one of the two variables \( s \) mentioned in Lemma 3. We put \( H = \frac{1}{2} \), and define \( d_a(n) \) by the formula
\[ (\zeta(s))^a = \sum_{n=1}^{\infty} \frac{d_a(n) n^{-s}}{n} \]
valid in \( \sigma > 2 \).

Next we write \( P(s) = \sum_{1 \leq n < H^*} (d_a(n) n^{-s}) \)
for all complex \( s \). We now compare
\[ f(\sigma) = \sum_{j} \int_{j} (\zeta(s + it))^P - (\zeta(s + it))^q \, dt \]
for \( \sigma = \sigma_1 = \frac{1}{2} + \frac{q}{\log H} \), \( \sigma = \sigma_2 = \frac{1}{2} + \frac{Dq}{\log H} \), and \( \sigma = \sigma_3 = 10 \), where \( D \) is suitable positive constant.
Lemma 4: We have,

\[ \left( \frac{1}{H} f(\sigma_2) \right)^{\sigma_3 - \sigma_1} \ll \left( \frac{1}{H} f(\sigma_1) \right)^{\sigma_3 - \sigma_2} \left( \frac{1}{H} f(\sigma_3) \right)^{\sigma_2 - \sigma_1} \]

where the asterisk indicates the modification in the intervals $J$ by removing intervals of length $\log H$ at both the ends of $J$.

Proof

Follows by arguments from which we deduced Theorem 4 from Theorem 3. We give some details. We take the kernel function $\phi(w) = \exp(w^{1002})$ and deduce the convexity for

\[ \frac{1}{(\log H)^A} \int_J \left| (\zeta(s))^p - (\zeta(s^q))^q \right|^q \, dt \]

From this result Lemma 4 follows by Holder’s inequality.

Lemma 5: If

\[ \max \left( \frac{1}{H} \sum_J \int_J | \zeta(\sigma_1 + it) |^{2a} \, dt, \right. \]

\[ \frac{1}{H} \sum_J \int_J | \zeta(\sigma_2 + it) |^{2a} \, dt \left) \right. = 0 \right( (\log H)^2 \right) q^k \right) k^2 \]

then for $\sigma = \sigma_1$ and $\sigma = \sigma_2$, we have,

\[ \frac{1}{H} f(\sigma) \gg \text{and} \ll \left( \sigma - \frac{1}{2} \right)^{-a^2} \]

by a suitable choice of the constants $A$ and $B$.

Proof:

First we remark that for any two real numbers $x, y,$
we have,

\[ \frac{2}{\| m - y \|^q} < 2^q (\| x \|^q + \| y \|^q) \]

and so

\[ \frac{2}{\| x - y \|^q} < 2^q (\| m \|^q + (2^q - 1) \| y \|^q) \]

From this and the hypothesis of the lemma, it follows that for \( \sigma = \sigma_1 \) and \( \sigma = \sigma_2 \) we have

\[ \frac{1}{H} \mathcal{L}(\sigma) \sim \frac{1}{H} \int J \int |P(\sigma + it)|^2 \, dt. \]

Here the LHS can be replaced by \( \frac{1}{H} \int T+H \int |P(\sigma + it)|^2 \, dt \) with an error

\[ \frac{1}{H} \int \int |P(\sigma + it)|^2 \, dt \]

\[ = O((\log H)^b \left( \frac{1}{H} \int T+H \int |P(\sigma + it)|^4 \, dt \right)^2), \]

where \( b = \frac{1}{2} (k + 2 - B_k) \). Now (by a theorem of Montgomery and Vaughan, for a simple proof of the result necessary here see [4]), we have

\[ \frac{1}{H} \int T+H \int |P(\sigma + it)|^2 \, dt \]

\[ = \sum_{n < H^2} \left( 1 + O\left( \frac{n}{H} \right) \right) \frac{(d_A(n))^2}{n^2\sigma}, \]
and

\[
\frac{1}{H} \int_0^T \frac{1}{|P(\sigma + it)|^4} \, dt
\]

\[
\ll \sum_{n<H} \left( 1 + \frac{n}{H} \right) \frac{(d_{2a}(n))^2}{n^{2\sigma}}.
\]

We now use the well-known results

\[
\frac{1}{x} \sum_{x < n < 2x} (d(n))^2 \gg (\log x)^2 - 1,
\]

and

\[
\frac{1}{x} \sum_{x < n < 2x} (d_{2a}(n))^2 \gg (\log x)^4 a^2 - 1,
\]

where the constants implied by \( \gg \) and \( \ll \) depend only on \( k \).

From these remarks lemma 5 follows by choosing \( A \) and \( B \) suitably.

Lemma 6

\[\left( \frac{1}{H} f(\sigma_3) \right) \gg \left( \sigma_3 - \frac{1}{2} \right)^{-a^2} \]

\[\left( \frac{1}{H} f(\sigma_1) \right) \ll \left( \sigma_1 - \frac{1}{2} \right)^{-a^2} \]

and

\[\left( \frac{1}{H} f(\sigma_3) \right) \ll H^{-100} q,\]

where the constants implied by \( \gg \) and \( \ll \) are independent of \( D \) and \( H \).

Proof

The coefficients in \((\zeta(s))^p - (P(s))^q\) of \( n^{-s} \) are zero
for \( 1 < n < H^2 \) and for \( n > H^2 \) they are trivially \( O(n^2 \zeta(2)) \). Hence the last assertion follows. The first assertion follows from the definition of \( \left( \frac{1}{H} f(\sigma_2) \right)_n \) provided \( A \) is chosen large. The second assertion follows from lemma 5.

From Lemmas 4 and 6, we have, by inserting the values of \( \sigma_1, \sigma_2, \sigma_3 \) the following lemma 7.

**Lemma 7.** We have,

\[
\frac{a^2}{qD} \left( 1 - \frac{q}{\log H} \right)
\ll \left( \frac{\log H}{q} \right) a^2 \left( 10 - \frac{Dq}{\log H} \right) \exp \left( - \frac{D}{50} \right),
\]

where the constant implied by \( \ll \) is independent of \( D \) and \( H \).

Since lemma 7 is false for a suitable constant \( D > 0 \), we have by lemma 3,

**Lemma 8.** We have,

\[
\max \sigma = \sigma_1 \sigma_2 \left( \frac{1}{H} \sum_j \int_j \zeta(\sigma + it) t^2 a \, dt \right) \gg \left( \frac{\log H}{q} \right) a^2,
\]

where \( \sigma_1 = \frac{1}{2} + \frac{q}{\log H} \), and \( \sigma_2 = \frac{1}{2} + \frac{Dq}{\log H} \).

Hence

\[
M \left( \frac{1}{2} \right) \gg (\log H)^{a^2} - a^2.
\]
Lemma 9. We have,

\[ M \left( \frac{1}{2} \right) \gg \left( \log H \right)^{k} q_{n} - k^{2}. \]

Proof

Follows from lemma 8 since \( a = \frac{p_{n}}{q_{n}^{2}}, q = q_{n} \),

\[ a^{2} = k^{2} + O \left( \left( k - \frac{p_{n}}{q_{n}} \right) \left( \frac{1}{q_{n} q_{n}^{'}} \right) \right) \]

and

\[ q_{n} q_{n}^{'} + 1 > \log \log H > q_{n} - 1 q_{n}. \]

Theorem 1 is completely proved by lemma 9.

§ 4. Some general remarks

It is possible to prove a very general theorem which is important for many applications. For example mean-value lower-bounds for \( | \zeta \left( \frac{1}{2} + it \right) |^{2k} \), \( k > 0 \) integer \( (k > 0 \) irrational and in this case we have to assume Riemann hypothesis to obtain optimal lower bounds), \( \Omega \) theorems and so on. This has already been done in [5]. The use of kernels like \( \text{Exp} \left( \frac{(\sin w)^{2}}{2} \right) \) enables us to state improvements of results in [5]. These are really corollaries to results in [5] (which look like improvements) if we use the theorem 3 due to Gabriel to be stated at the end of this section. However we can also avoid Gabriel's theorem and prove the corollaries by repeating the arguments of [5], replacing the old kernels \( \text{Exp} \left( \frac{4a+2}{w} \right), a > 0 \) integer, by the new kernels. Here we show how to deduce from the results of [5] corollaries which look like improvements. For this purpose we recall definition of Titchmarsh series with a modification.

Titchmarsh series

Let \( A \geq 1 \) be a constant. Let \( 1 = \lambda_{1} < \lambda_{2} < \lambda_{3} \ldots \).
where \( \frac{1}{A} < \lambda_{n+1} - \lambda_n < A \). Next let \( a_1, a_2, a_3, \ldots \) be a sequence of complex numbers possibly depending on a parameter \( H > 0 \), such that \( a_1 = 1 \), and \( |a_n| < (nH_0)^A \) where \( H_0 = H + 1000 \). Put \( F(s) = \sum_{n=1}^{\infty} \left( a_n \lambda_n^{-s} \right) \) where \( s = \sigma + it \). Surely \( F(s) \) is absolutely convergent in \( \sigma > A + 2 \). \( F(s) \) is called a Titchmarsh series if there exists a constant \( A > 1 \) and a system of rectangles \( R(T, T+H) \) defined by \( (\sigma > 0, T < t < T + H) \) where \( 0 < H < T \) and \( T \) (which may be related to \( H \)) tends to infinity and \( F(s) \) admits an analytic continuation into these rectangles and is continuous on the left boundary \( L \) of \( R(T, T+H) \) namely \( \sigma = 0, T < t < T + H \).

I now state a conjecture.

Conjecture

We have, for a Titchmarsh series \( F(s) \),

\[
\max_{\sigma > 0} \left( \frac{1}{H} \int_{L} |F(\sigma + it)|^2 \, dt \right) > C_A \sum_{n < X} |a_n|^2. 
\]

where \( X = 1 + D_A H \), and \( C_A \) and \( D_A \) are positive constants depending only on \( A \).

The progress in the direction of this conjecture can be stated as a theorem.

Theorem 5

We have, for a Titchmarsh series \( F(s) \),

\[
\max_{\sigma > 0} \left( \frac{1}{H} \int_{L} |F(\sigma + it)|^2 \, dt \right) > C_A \sum_{n < X} |a_n|^2 \left( 1 - \frac{\log n}{\log H_0} + \frac{1}{\log \log H_0} \right).
\]
where $C_A$, $D_A$ and $X$ have the same notation as in the conjecture.

Further $C_A$ and $D_A$ are effective. We have also,

$$\max_{\sigma \geq 0} \left( \frac{1}{H} \int_{L} |F(\sigma + it)| \, dt \right) > C_A.$$ 

Remark

Since we have $|F(\sigma + it)| > \frac{1}{2}$ for

$$\sigma > 100A \left( \frac{\log H_0}{2} \right)^2,$$

we can assume without loss of generality that $H$ exceeds a constant depending only on $A$.

In [5] I proved (note that $\lambda_n \gg$ and $\ll n$).

Theorem 6

We have, for a Titchmarsh series $F(s)$,

$$\frac{1}{H} \int_{L} |F(it)|^2 \, dt > C_A \sum_{n < X} |a_n|^2 \left( 1 - \frac{\log n}{\log H_0} + \frac{1}{\log \log H_0} \right),$$

provided the maximum of $|F(s)|$ taken over $R(T, T+H')$, does not exceed $\text{Exp} \left( \frac{H_0}{100A} \right)$. We have, also

$$\frac{1}{H} \int_{L} |F(it)| \, dt > C_A.$$

As already stated we deduce Theorem 5 from Theorem 6. (In fact for this deduction even a milder version of Theorem 6 namely

$$\frac{1}{H} \int_{0}^{H} \left| \sum_{n < \text{Exp} \left( (\log H_0)^2 \right)} a_n \lambda_n^{-it} \right|^2 \, dt > \frac{1}{n \log \log H_0}$$

$$C_A \sum_{n < X} |a_n|^2 \left( 1 - \frac{\log n}{\log H_0} + \frac{1}{\log \log H_0} \right),$$
or even the result with \((\log H_0 \log \log H_0)^2\) in place of 
\((\log H_0)^2\) is enough). From Theorem 5 we deduce the 
following Theorem 7 using the new kernel and Theorem 3' 
below. As remarked already we can prove Theorems 5 and 
7 directly repeating the arguments of [5] with the new kernel.

**Theorem 7**

The assertions of Theorem 6 are true even with the assumption 
that the maximum of \(|F(s)|\) taken over \(R(T, T+H)\) 
does not exceed \(\exp \exp \left(\frac{1}{(10A)^4H_0}\right)\)

**Remark 1**

Theorem 5 is more fundamental. Given better kernels 
we can deduce better Theorems than Theorem 7.

**Remark 2**

We can state many corollaries to Theorem 7. We mention 
some of them here. The first is that 
\[
\frac{1}{H} \int_{T}^{T+H} \left| \sum_{n} \left(\frac{d_k(n)}{n}\right)^2 n^{-1} \left(\frac{C \log \log H}{100}\right)^{-1} \right|^2 \ dt
\]

is \(\gg (\log H)^k\) for all integers \(k > 0\). The second is the 
validity of the same result for all irrational \(k > 0\) on the 
assumption of Riemann hypothesis. In both these cases we 
need the condition \(T > H \gg \log \log T\), and the constant 
implied by \(\gg\) is independent of \(T\) and \(H\) and is supposed 
to be large enough. The next is

\[
\max_{T < t < T + H} |\zeta(\frac{1}{2} + it)| >
\]

\[
\left( \sum_{n < \frac{H}{100}} \left(\frac{d_k(n)}{n}\right)^2 n^{-1} \left(\frac{C \log \log H}{100}\right)^{-1} \right)^\frac{1}{2k}
\]

where \(k > 1\) is any integer subject to \(3^k < H\) and \(C (> 0)\) 
is independent of \(k\), \(T\) and \(H\). (We still need the condition
T > H \gg \log \log T). By an optimal choice of k the RHS here was shown by R. Balasubramanian (Ref. On the frequency of Titchmarsh phenomenon for \( \zeta(s) - 1 \), J. of Number theory, to appear) to exceed
\[
\exp \left( \frac{3}{4} \left( \frac{\log H}{\log \log H} \right)^{\frac{3}{2}} \right).
\]
It also gives similar results for
\[
\max | \zeta(1 + it) | \quad \text{and} \quad \max | \zeta(\sigma + it) | \quad \text{where} \quad \frac{1}{2} < \sigma < 1.
\]
The results for \( \max | \zeta(\frac{1}{2} + it) | \) and \( \max | \zeta(1 + it) | \) have some perfection which is not yet available for \( \frac{1}{2} < \sigma < 1 \). We expect
\[
\max | \zeta(\sigma + it) | > \exp \left( C_1 \frac{(\log H)^{1-\sigma}}{(\log \log H)^{\sigma}} \right), \quad C_1 > 0
\]
a constant independent of T and H, but we have such a result with \( \log \log H \) in place of \( (\log \log H)^{\sigma} \).

We now deduce Theorem 5 from Theorem 6. As remarked already we can suppose that H exceeds a large constant depending on A. Suppose Theorem 5 is false. Then by the fact that the mean-value of the modulus of an analytic function over a disc is not less than its value at the centre, we see that the maximum of \(| F(s) | \) taken over \( \sigma > \frac{2}{\log H} \),

\( T + 1 < t < T + H - 1 \) does not exceed \( C_A H^{4A} \). (We can now consider the series
\[
\sum_{n=1}^{\infty} \left( a_n \lambda_n \right) \exp \left( -\frac{\lambda_n}{Y} \right)
\]
with for example \( Y = \exp \left( \frac{1}{4} \log H \log \log \log H \right) \). Let \( 0 < \sigma_1 < \sigma_2 \). It is not hard to show that the absolute value of this series at \( s = s_2 = \sigma + it \) does not exceed
\[
| F(s_2) | + \frac{1}{2\pi} \int \left| F(s_2 + w) \right| Y^{\text{Re}(w)} \Gamma(w + 1) \left| \frac{d w}{w} \right|, \quad \text{Re} w = \sigma_1 - \sigma_2, \quad | \Im w - t | < \frac{1}{2} H
\]
plus a negligible quantity. We may use things like \( \int \left| \frac{dw}{w} \right| \).

the integral taken over \( |\text{Im} (w)| < 1 \) is \( O \left( (\log |\text{Re} (w)| )^{-1} \right) \).

Now by applying Theorem 6 to \( F(s) \) (or its truncation), it follows that

\[
\max \sigma > \frac{2}{\log H} \left( \frac{1}{H-2} \int_{T+1}^{T+H-1} |F(\sigma + it)|^2 \, dt \right)
\]

\[
> e^{-40C_A} \sum_{n<\alpha} |a_n|^2 \left( 1 - \frac{\log n}{\log H} + \frac{1}{\log \log H} \right)
\]

Hence Theorem 5 is proved with \( e^{-40C_A} \) in place of \( C_A \).

Next we deduce Theorem 7 from 5. For this purpose we need the following Theorem 3' from which Gabriel deduced his Theorem 3 stated in § 2.

Theorem 3' (R. M. Gabriel)

Let \( R \) be the closed rectangle with vertices \( z_0, z_0, -z_0, -z_0 \). Let \( F(z) \) be continuous on the boundary of \( R \) and regular in the interior of \( R \). Then

\[
\int_L \left| F(z) \right|^q \, |dz| \leq \left\{ \int_{P_1} \left| F(z) \right|^q \, |dz| \right\}^{\frac{1}{q}} \left\{ \int_{P_2} \left| F(z) \right|^q \, |dz| \right\}^{1-q}
\]

for any \( q^* > 0 \), where \( L \) is the line segment from \( \frac{1}{2} (z_0 - z_0) \) to \( \frac{1}{2} (z_0 - \overline{z_0}) \), \( P_1 \) consists of three line segments connecting \( \frac{1}{2} (z_0 - \overline{z_0}), \overline{z_0}, z_0 \) and \( \frac{1}{2} (z_0 - \overline{z_0}) \); and \( P_2 \) is the mirror image of \( P_1 \) in \( L \).

Let \( \sigma_0 \) be the abscissa at which the maximum is attained.
Now $0 < \sigma_0 < \sigma > A + 2$, we have,

$$\frac{1}{H} \int_{T}^{T+H} F(\sigma + it) |^2 \, dt$$

$$= \sum_{n=1}^{\infty} |a_n|^2 \left( 1 + O\left( \frac{n}{H} \right) \right) n^{-2\sigma},$$

where the $O$-constant is absolute. We now choose $F(z)$ in Theorem 3' to be $G(s)$ defined by,

$$G(s) = F\left( s + \sigma_0 + \frac{iH}{2} \right) \exp\left( \left( \sin\left( \frac{w - s - \sigma_0 - \frac{iH}{2}}{2A + 4} \right) \right)^2 \right),$$

where we have written $s$ for $z$. We now choose $z_0 = s_0 = \sigma_0 + \frac{iH}{4}$, and $R$, $L$, $P_1$, $P_2$ accordingly as in Theorem 3'. We write $w = \zeta v$ and average the inequality provided by Theorem 3' with respect to $v$ from $v = T - \frac{iH}{2}$ to $T + \frac{3H}{2}$. Theorem 7 follows since the horizontal sides in $P_1$ and $P_2$ contribute a negligible quantity by our assumption on the size of $|F(s)|$. The details are as follows. We have,

$$\frac{1}{2H} \int_{W}^{L} \int_{P_1} |G(s)| \, dv \, dt < \left( \frac{1}{2H} \int_{W}^{L} \int_{P_1} |G(s)| \, dv \, dt \right)^{\frac{1}{2}}$$

$$+ \frac{1}{2H} \int_{W}^{L} \int_{P_2} |G(s)| \, dv \, dt$$

where $P_1$ and $P_2$ denote the vertical portions of $P_1$ and $P_2$. Our choice of $\sigma_0$ poses a slight difficulty. To avoid this we choose $\sigma_0$ such that LHS here is maximum. This proves Theorem 7, since the main term on the RHS is now,
\[
< \left( \frac{1}{2H} \int_0^1 \int \frac{1}{p_2} |G(s)| \, dv \, dt \right)^{1/2}
\]

\[
\left( \frac{1}{2H} \int_0^1 \int \frac{1}{L} |G(s)| \, dv \, dt \right)^{1/2}
\]

**Remark**

The result

\[
\frac{1}{T} \int_T^{2T} \zeta \left( \frac{1}{2} + it \right) \zeta' \left( \frac{1}{2} + it \right) \frac{2k}{\zeta \left( \frac{1}{2} + it \right)} \, dt > C_k \left( \log T \right)^{k^2},
\]

for irrational \( k > 0 \) seems to be difficult. However using the theorem that every real irrational is the sum of two real numbers each of which have partial quotients \( < 4 \) in their simple continued fraction expansions one may hope to get some day the lower bound \( C_k \left( \log T \right)^{k^2} \left( \log \log T \right)^{-\frac{1}{2}} k^2 \).

**The theorem on the decomposition into two real numbers referred to here is due to Marshall Hall Jr. (Ref. Annals of Maths. 48 (1947), 966-993). For a simple proof of a weaker result see J. W. S. Cassels, Mathematika 3 (1956), 109-110.**

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