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MEAN-VALUE OF THE RIEMANN ZETA-FUNCTION AND OTHER REMARKS-III

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§ 1. Introduction

Regarding the mean value lower bounds for $\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}$, Titchmarsh was the first to prove (see Theorem 29 on p.42 of [7]) that

$$\frac{1}{T}\int_{T}^{2T}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k}dt > C_k(\log T)^{k^2}, \quad (k>0), \quad (1)$$

where $T \geqslant 100$ and k *is positive integer* and $C_k > 0$ depends only on k. (However, Theorem 29 is stated without proof in [6], and the reference in [6] to published papers refering to Theorem 29 indicate that he proved $\limsup \frac{\text{LHS}}{\text{RHS}} > 0$ as $T \to \infty$). In many of my recent papers ([2, I, II], [3, I, II]) I considered extension of (1) to non-integral values of k and more general questions. In particular I proved that (1) holds for all $k > 0$ for which $2k$ is *an integer*. (I proved also that (1) holds for all $k > 0$ *on Riemann hypothesis*). My proof of this was self-contained and did not make use of Gabriel's convexity theorem. Heath-Brown noticed [1] that the use of Gabriel's theorem (for reference to Gabriel's theorem see [I]) would not only simplify my proof of (1) for integral k and half an odd integer, but would yeild further dividends like the proof of (1) for all rational k. In my papers [2, I] and [3, II], I dealt with the case k irrational and proved for the LHS of (1) the lower bound $C_k\left(\frac{\log T}{\log\log T}\right)^{k^2}$. It is the purpose of this note to give a proof of a more illuminating lower bound $C_k\left(\frac{\log T}{q_n}\right)^{k^2}$ (to be made precise) which is slightly a better lower bound. It is still very far from the lower bound $C_k (\log T)^{k^2}$. The present lower bound is made possible by Heath-Brown's idea [1] of using Gabriel's theorem to prove (1) for all rational k. As in our earlier papers we start with the fundamental function

$M(\alpha) = M(\alpha, k, T, T+H) =$

$$\max_{\sigma > \alpha} \left(\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \right), \quad (2)$$

where $T > H > 0$, $\alpha > \frac{1}{2}$ and $k > 0$. We prove the following (Hereafter we suppose $k > 0$ and k is irrational) theorem.

Theorem 1 :

Let $H_0 = H + 10000$ *Then*

$$M\left(\frac{1}{2}\right) > C_k \left(\frac{\log H_0}{q_n}\right)^{k^2} \quad (3)$$

where $C_k > 0$ *depends only on* k. *Here* $\frac{p_m}{q_m}$ *is the* m^{th} *convergent to the simple continued fraction expansion of k and* n *is the unique integer such that* $q_n q_{n+1} > \log\log H_0 > q_{n-1} q_n$.

From this it is possible to deduce (as a very general principle applicable in many situations) with the help of Gabriel's theorem (using a suitable kernel function) the following theorem (See § 2)

Theorem 2 :

If $T > H > 10000 \log\log T > 10^8$, *then we have,*

$$\frac{1}{H}\int_{T}^{T+H} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt > C_k \left(\frac{\log H}{q_n}\right)^{k^2}, \tag{4}$$

where C_k *and* q_n *are as in theorem 1.*

Remark 1 : Since $\log\log H_0 > q_n q_{n-1}$ we can replace the RHS in (4) by $C_k \left(\frac{q_{n-1} \log H}{\log\log H_0}\right)^{k^2}$. However q_{n-1} cannot in general be replaced by a more explicit function of H_0.

Remark 2 If the s. c. f expansion of k has bounded partial quotients, then it is well-known that q_n is around $(\log\log H_0)^{\frac{1}{2}}$ and hence we can replace the RHS in (4) by $C_k (\log H_0)^{k^2} (\log\log H_0)^{-\frac{1}{2}k^2}$ for such k. By an extension of this argument we note that (by using a deep theorem of K. F. Roth [6]) that we can replace the RHS of (4) by $C_k (\log H_0)^{k^2} (\log\log H_0)^{-\frac{1}{2}k^2 - \varepsilon}$, where $\varepsilon > 0$, $C_{k,\varepsilon} > 0$ provided k is algebraic. Roth's result however renders C_k to be an *ineffective constant* for all small constants $\varepsilon > 0$.

Remark 3. Theorems 1 and 2 have their extensions to L-series and hybrid analogues and we do not bother to state them here.

Remark 4. As remarked already Theorem 2 shows that

$$\lim_{T\to\infty}\left(\frac{\psi(T)}{T(\log T)^{k^2}}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt\right)$$ is infinity if

$\psi(T) = (\log\log T)^{k^2}$. Theorem 2 also shows that if

$\psi(T) = (\log\log T)^{k^2/2}$ then $\limsup_{T\to\infty}(\ldots) > 0$. Further if

$\psi(T)$ is any function which tends to infinity as T then there exists an irrational k (depending on the nature of the function ψ) such that $\limsup_{T\to\infty}(\ldots) = \infty$.

Remark 5 : For any rational number r in its lowest terms, let $H(r)$ denote the sum of the absolute values of the numerator and the denominator and 100. Write $L_1(x) = \log x$, $L_2(x) = \log\log x$ and so on Consider the intervals

$[r - \delta(r), r + \delta(r)]$, where $\delta(r) = \varepsilon(H(r))^{-2}$ $(L_1(H(r)))^{-1}(L_2(H(r)))^{-2}$ where ε is any

constant satisfying $0 < \varepsilon < \frac{1}{1000}$. The sum of the lengths of these intervals is $\leq 100\,\varepsilon$ (as can be easily seen by taking all rationals r with $H(r)$ lying in $[2^m, 2^{m+1})$, $m = 1, 2, 3, \ldots$). This observation is due to Khintchine and gives following corollary to Theorem 1 and 2 Namely we can replace the RHS in (3) and (4) by

$$C_k\,(L_1(H_0)(L_4(H_0))^{-1})^{k^2}\,(L_2(H_0)\,L_3(H_0))^{-\frac{k^2}{2}}$$

for almost all k. This corollary was pointed out to me by Dr. M Ram Murthy.

For some more results see § 4. These deal with Titchmarsh series.

§ 2. Deduction of Theorem 2 From Theorem 1

Theorem 1 is more fundamental. Depending on the availability of bounds for and suitable kernels we can deduce Theorem 2 from Theorem 1, with the help of Gabriel's Theorem viz

Theorem 3

(R. M GABRIEL) *Let $f(z)$ be regular in the infinite strip $\alpha < \text{Re } z < \beta$ and continuous in $\alpha \leq \text{Re } z \leq \beta$. Suppose $|f(z)| \to 0$ uniformly as $|z| \to \infty$ in $\alpha \leq \text{Re } z \leq \beta$. Then for any $q^* > 0$, we have.*

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^{q^*} dt \leq$$

$$\left(\int_{-\infty}^{\infty} |f(\alpha + it)|^{q^*} dt\right)^{\lambda} \left(\int_{-\infty}^{\infty} |f(\beta + it)|^{q^*} dt\right)^{\mu},$$

where $\lambda = \dfrac{\beta - \gamma}{\beta - \alpha}$ *and* $\mu = \dfrac{\gamma - \alpha}{\beta - \alpha}$

(*Note that the integrals are not required to converge If the* RHS *is finite then so is* LHS).

Later in § 4 we state Theorem 3′ (also due to Gabriel) from which he deduced Theorem 3. Both the theorems 3 and 3′ were used by Heath-Brown in [1]

Remark : If $\phi(w)$ is a suitable kernel function (which is analytic) then replacing $|f(z)|^{q^*}$ by $|f(z)\,\phi(w - z)|^{q^*}$ we can replace the three integrals $|f(x + it)\,\phi(iv - x - it)|^{q^*}$ with $x = \gamma$, α and β respectively. This would mean (in a certain sense) a localisation in a short neighbourhood of ν (the shortness depending on the fastness of tapering of the function ϕ (w)). Hence if we now average both sides of the inequality with respect to ν over a short interval we get a convexity result for the mean-value of $|f(x+it)|^{q^*}$ ($x = \gamma$, α and β) as x varies over a short interval. Thus we state the following theorem.

Theoram 4 :

We have, by taking $f(z) = \zeta(z) - \frac{1}{z-1}$ *and* $\phi(w) = \operatorname{Exp}\left\{\left(\sin\left(\frac{w}{100}\right)\right)^2\right\}$, $\alpha = \frac{1}{2}$, $\beta = 2$ *and* γ *any number in* $[\alpha, \beta]$, *we have,*

$$\frac{1}{H}\int_{T}^{T+H} |\zeta(\gamma + it)|^{q^*} dt \ll$$

$$\left(\frac{1}{H}\int_{T-\frac{H}{10}}^{T+H+\frac{H}{10}} |\zeta(\alpha + it)|^{q^*} dt\right)^{\lambda} \left(\frac{1}{H}\int_{T-\frac{H}{10}}^{T+H+\frac{H}{10}} |\zeta(\beta+it)|^{q^*} dt\right)^{\mu}$$

where $T > H > 10000 \log\log T > 10^8$, *and the constant implied by* $\ll$ *depends only on* q^*.

From Theorem 4 we can easily deduce Theorem 2 as a corollary to Theorem 1.

§ 3 **Proof of Theorem 1.** We give a brief sketch. We can assume without loss of generality that H exceeds a sufficiently large constant since $|\zeta(10 + it)| > \frac{1}{2}$. We can now write H for H_0.

Lemma 1 *In* ($\sigma > \frac{1}{2} + \frac{1}{\log H}$, $T+1 \leq t \leq T+H-1$)

we divide the t-range into intervals I *of length* $(\log H)^{A}$ *(where* $A > 0$ *is a suitable constant) ignoring a bit at one end if necessary. Denote the maximum of* $|\zeta(s)|$ *in* $\sigma \geq \frac{1}{2} + \frac{1}{\log H}$, *t in* I *by* $\mu(I)$. *Then*

$$\sum_{I} (\mu(I))^{2k} \ll H\, M(\tfrac{1}{2})\, (\log H)^{2}.$$

Remark : Hereafter we assume that $M(\frac{1}{2}) \ll (\log H)^{k^2}$ since otherwise there is nothing to prove. Hence the LHS is $\ll (\log H)^{k^2+2}$

Proof : Follows from the fact that $|\zeta(s)|^{2k}$ does not exceed its mean-value over a disc with centre s and radius $\ll (\log H)^{-1}$

Lemma 2 : *Let* $B > 0$ *be a constant. Then the number of intervals* I *for which* $\mu(I) > (\log H)^{B}$, *is not more than* $H(\log H)^{k^2+2-2Bk}$. *Denote any of the remaining intervals by* J. *Then*

$$\sum_{J}\int_{J} |\zeta(s)|^{2k}\, dt \gg \sum_{J}\left(\int_{J} |\zeta(s)|^{2a}\, dt + O(\log H)^{A}\right),$$

where $p = p_n$, $q = q_n$, $a = \frac{p}{q}$ *and* $s = \frac{1}{2} + \frac{q}{\log H} + it$.

Proof : The first part follows from Lemma 1. The second part follows from the remark

$$\int_{J} |\zeta(s)|^{2k}\, dt \gg \int_{J,\, |\zeta(s)| \geq 1} |\zeta(s)|^{2a}\, dt$$

$$\geq \int_{J} |\zeta(s)|^{2a} - (\log H)^{A}$$

which is valid since, if $(\log H)^{B} \geq |\zeta(s)| \geq 1$, then $|\zeta(s)|^{2k-2a} = \operatorname{Exp}((2k-2a)\log|\zeta(s)|) \geq \operatorname{Exp}(-2B)$.

(We have to use the well-known result

$| k - a | < \frac{1}{q_n q_{n+1}} \left(< \frac{1}{\log \log H} \right)$ from the theory of continued fractions).

Lemma 3 *We have, with* $s = \frac{1}{2} + \frac{q}{\log H} + it$,

$$\sum_J \int_J |\zeta(s)|^{2k} dt \gg \sum_J \int_J |\zeta(s)|^{2a} dt + O(H)$$

where $a = \frac{p_n}{q_n}$, *and* $q = q_n$. *The same inequality is also valid if* $s = \frac{1}{2} + \frac{Dq}{\log H} + it$ *where* $D > 0$ *is a constant.*

Proof : Follows from Lemma 2.

The rest of the work consists in proving that if D is large enough the RHS in the inequality of Lamma 3 is $\gg H (\log H)^{k^2} q^{-k^2}$, for at least one of the two variables s mentioned in Lemma 3. We put $H^* = H^{\frac{1}{2}}$, and define $d_a(n)$ by the formula $(\zeta(s))^a = \sum_{n=1}^{\infty} (d_a(n) n^{-s})$, valid in $\sigma > 2$.

Next we write $P(s) = \sum_{1 \le n \le H^*} (d_a(n) n^{-s})$

for all complex s. We now compare

$$f(\sigma) = \sum_J \int_J | (\zeta(\sigma + it))^p - (P(\sigma + it))^q |^{\frac{2}{q}} dt$$

for $\sigma = \sigma_1 = \frac{1}{2} + \frac{q}{\log H}$, $\sigma = \sigma_2 = \frac{1}{2} + \frac{Dq}{\log H}$, and $\sigma = \sigma_3 = 10$, where D is suitable positive constant.

Lemma 4 : *We have,*

$$\left(\frac{1}{H} f(\sigma_2)\right)_*^{\sigma_3 - \sigma_1} \ll \left(\frac{1}{H} f(\sigma_1)\right)^{\sigma_3 - \sigma_2} \left(\frac{1}{H} f(\sigma_3)\right)^{\sigma_2 - \sigma_1},$$

where the asterisk indicates the modification in the intervals J *by removing intervals of length* log H *at both the ends of* J

Proof

Follows by arguments from which we deduced Theorem 4 from Theorem 3. We give some details. We take the kernel function $\phi(w) = \text{Exp}(w^{1002})$ and deduce the convexity for

$$\frac{1}{(\log H)^A} \int_J \left| (\zeta(s))^p - (P(s))^q \right|^{\frac{2}{q}} dt$$

From this result Lemma 4 follows by Holder's inequality.

Lemma 5 : *If*

$$\max\left(\frac{1}{H}\sum_J \int_J |\zeta(\sigma_1 + it)|^{2a} dt,\right.$$

$$\left.\frac{1}{H}\sum_J \int_J |\zeta(\sigma_2 + it)|^{2a} dt\right) = O\left((\log H)^{k^2} q^{-k^2}\right),$$

then for $\sigma = \sigma_1$ *and* $\sigma = \sigma_2$, *we have,*

$$\frac{1}{H} f(\sigma) \gg \text{ and } \ll \left(\sigma - \frac{1}{2}\right)^{-a^2},$$

by a suitable choice of the constants A *and* B.

Proof :

First we remark that for any two real numbers $x, y,$

we have,

$$|x-y|^{\frac{2}{q}} < 2^{\frac{2}{q}} \left(|x|^{\frac{2}{q}} + |y|^{\frac{2}{q}} \right)$$

and so

$$|x-y|^{\frac{2}{q}} - |y|^{\frac{2}{q}} < 2^{\frac{2}{q}} |x|^{\frac{2}{q}} + \left(2^{\frac{2}{q}} - 1 \right) |y|^{\frac{2}{q}}$$

From this and the hypothesis of the lemma, it follows that for $\sigma = \sigma_1$ and $\sigma = \sigma_2$ we have

$$\frac{1}{H} f(\sigma) \sim \frac{1}{H} \sum_{J} \int_{J} |P(\sigma+it)|^2 \, dt.$$

Here the LHS can be replaced by $\frac{1}{H} \int_{T}^{T+H} |P(\sigma+it)|^2 \, dt$,

with an error

$$\frac{1}{H} \sum_{I \neq J} \int_{I} |P(\sigma+it)|^2 \, dt$$

$$= O\left((\log H)^b \left(\frac{1}{H} \int_{T}^{T+H} |P(\sigma+it)|^4 dt \right)^{\frac{1}{2}} ,\right)$$

where $b = \frac{1}{2}(k^2 + 2 - Bk)$. Now (by a theorem of Montgomery and Vaughan, for a simple proof of the result necessary here see [4]), we have

$$\frac{1}{H} \int_{T}^{T+H} |P(\sigma+it)|^2 dt$$

$$= \sum_{n < H^*} \left(1 + O\left(\frac{n}{H} \right) \right) \frac{(d_a(n))^2}{n^{2\sigma}},$$

and

$$\frac{1}{H}\int_{T}^{T+H} |P(\sigma+it)|^{4} dt$$

$$\ll \sum_{n<H}\left(1+\frac{n}{H}\right)\frac{(d_{2a}(n))^{2}}{n^{2\sigma}}.$$

We now use the well-known results

$$\frac{1}{x}\sum_{x<n<2x}(d_{a}(n))^{2} \gg \text{ and } \ll (\log x)^{a^{2}-1},$$

and

$$\frac{1}{x}\sum_{x<n<2x}(d_{2a}(n))^{2} \gg \text{ and } \ll (\log x)^{4a^{2}-1},$$

where the constants implied by $\gg$ and $\ll$ depend only on k. From these remarks lemma 5 follows by choosing A and B suitably.

Lemma 6

We have,

$$\left(\frac{1}{H} f(\sigma_2)\right) \gg \left(\sigma_2-\frac{1}{2}\right)^{-a^{2}},$$

$$\left(\frac{1}{H} f(\sigma_1)\right) \ll \left(\sigma_1-\frac{1}{2}\right)^{-a^{2}}$$

and

$$\left(\frac{1}{H} f(\sigma_3)\right) \ll H^{-\frac{1}{100\,q}},$$

where the constants implied by $\gg$ *and* $\ll$ *are independent of* D *and* H.

Proof

The coefficients in $(\zeta(s))^{p}-(P(s))^{q}$ of n^{-s} are zero for $1 < n < H^{\frac{1}{2}}$ and for $n > H^{\frac{1}{2}}$ they are trivially $O(n^2 (\zeta(2))^P.)$ Hence the last assertion follows. The first assertion follows from the definition of $\left(\frac{1}{H} f(\sigma_2)\right)_*$ provided A is chosen large. The second assertion follows from lemma 5.

From Lemmas 4 and 6, we have, by inserting the values of $\sigma_1, \sigma_2, \sigma_3$ the following lemma 7.

Lemma 7. *We have,*

$$\left(\frac{\log H}{qD}\right)^{a^2\left(1-\frac{q}{\log H}\right)} \ll \left(\frac{\log H}{q}\right)^{a^2\left(10-\frac{Dq}{\log H}\right)} \operatorname{Exp}\left(-\frac{D}{50}\right),$$

where the constant implied by $\ll$ *is independent of* D *and* **H**.

Since lemma 7 is false for a suitable constant $D > 0$, we have by lemma 5,

Lemma 8. *We have,*

$$\max_{\sigma=\sigma_1,\sigma_2}\left(\frac{1}{H}\sum_J\int_J |\zeta(\sigma+it)|^{2a}\,dt\right) \gg \left(\frac{\log H}{q}\right)^{a^2},$$

where $\sigma_1 = \frac{1}{2} + \frac{q}{\log H}$, *and* $\sigma_2 = \frac{1}{2} + \frac{Dq}{\log H}$.

Hence

$$M\left(\frac{1}{2}\right) \gg (\log H)^{a^2} q^{-a^2}.$$

Lemma 9. *We have,*

$$M(\tfrac{1}{2}) \gg (\log H)^{k^2} q_n^{-k^2}.$$

Proof

Follows from lemma 8 since $a = \frac{p_n}{q_n}$, $q = q_n$,

$a^2 = k^2 + O\left(\left(k - \frac{p_n}{q_n}\right)\right)$, $|k-a| < \frac{1}{q_n q_{n+1}}$ and

$q_n q_{n+1} > \log\log H > q_{n-1} q_n$.

Theorem 1 is completely proved by lemma 9.

§ 4. Some general remarks

It is possible to prove a very general theorem which is important for many applications. For example mean-value lower-bounds for $|\zeta(\frac{1}{2}+it)|^{2k}$, $k > 0$ integer ($k > 0$ irrational and in this case we have to assume Riemann hypothesis to obtain optimal lower bounds), Ω theorems and so on. This has already been done in [5]. The use of kernels like $\text{Exp}((\sin w)^2)$ enables us to state improvements of results in [5]. These are really corollaries to results in [5] (which look like improvements) if we use the theorem 3' due to Gabriel to be stated at the end of this section. However we can also avoid Gabriel's theorem and prove the corollaries by repeating the arguments of [5], replacing the old kernels $\text{Exp}(w^{4a+2})$, $a > 0$ integer, by the new kernels. Here we show how to deduce from the results of [5] corollaries which look like improvements. For this purpose we recall the definition of *Titchmarsh series with a modification.*

Titchmarsh series

Let $A \geq 1$ *be a constant. Let* $1 = \lambda_1 < \lambda_2 < \lambda_3 < \ldots,$

where $\frac{1}{A} < \lambda_{n+1} - \lambda_n < A$. *Next let* $a_1, a_2, a_3, \ldots$ *be a sequence of complex numbers possibly depending on a parameter* $H > 0$, *such that* $a_1 = 1$, *and* $|a_n| < (nH_0)^A$ *where* $H_0 = H + 1000$. *Put* $F(s) = \sum_{n=1}^{\infty} (a_n \lambda_n^{-s})$ *where* $s = \sigma + it$. *Surely* $F(s)$ *is absolutely convergent in* $\sigma > A + 2$. $F(s)$ *is called a Titchmarsh series if there exists a constant* $A > 1$ *and a system of rectangles* $R(T, T+H)$ *defined by* $(\sigma > 0, T < t < T + H)$ *where* $0 < H < T$ *and* T *(which may be related to* H*) tends to infinity and* $F(s)$ *admits an analytic continuation into these rectangles and is continuous on the left boundary* L *of* $R(T, T+H)$ *namely* $(\sigma = 0, T < t < T + H)$.

I now state a conjecture.

Conjecture

We have, for a Titchmarsh series $F(s)$,

$$\max_{\sigma > 0} \left(\frac{1}{H} \int_L |F(\sigma + it)|^2 dt \right) > C_A \sum_{n < X} |a_n|^2,$$

where $X = 1 + D_A H$, *and* C_A *and* D_A *are positive constants depending only on* A.

The progress in the direction of this conjecture can be stated as a theorem.

Theorem 5

We have, for a Titchmarsh series $F(s)$,

$$\max_{\sigma > 0} \left(\frac{1}{H} \int_L |F(\sigma + it)|^2 dt \right) >$$

$$C_A \sum_{n \leq X} |a_n|^2 \left(1 - \frac{\log n}{\log H_0} + \frac{1}{\log \log H_0} \right).$$

where C_A, D_A *and* X *have the same notation as in the conjecture. Further* C_A *and* D_A *are effective. We have also,*

$$\max_{\sigma>0}\left(\frac{1}{H}\int_L |F(\sigma+it)|\,dt\right) > C_A .$$

Remark

Since we have $|F(\sigma+it)| > \frac{1}{2}$ for $\sigma > 100\,A\,(\log H_0)^2$, we can assume without loss of generality that H exceeds a constant depending only on A.

In [5] I proved (note that λ_n is $\gg$ and $\ll n$),

Theorem 6

We have, for a Titchmarsh series $F(s)$,

$$\frac{1}{H}\int_L |F(it)|^2\,dt >$$

$$C_A \sum_{n\leq X} |a_n|^2 \left(1 - \frac{\log n}{\log H_0} + \frac{1}{\log\log H_0}\right),$$

provided the maximum of $|F(s)|$ *taken over* $R(T, T+H)$, *does not exceed* $\mathrm{Exp}\,(H_0^{100A})$. *We have, also*

$$\frac{1}{H}\int_L |F(it)|\,dt > C_A .$$

As already stated we deduce Theorem 5 from Theorem 6. (In fact for this deduction even a milder version of Theorem 6 namely

$$\frac{1}{H}\int_0^H \left|\sum_{n\leq \mathrm{Exp}((\log H_0)^2)} a_n\lambda_n^{-it}\right|^2 dt >$$

$$C_A \sum_{n\leq X} |a_n|^2 \left(1 - \frac{\log n}{\log H_0} + \frac{1}{\log\log H_0}\right),$$

or even the result with $(\log H_0 \log\log H_0)$ in place of $(\log H_0)^2$ is enough). From Theorem 5 we deduce the following Theorem 7 usi g the new kernel and Theorem 3' below. As remarked already we can prove Theorems 5 and 7 directly repeating the arguments of [5] with the new kernel.

Theorem 7

The assertions of Theorem 6 are true even with the assumption that the maximum of $|F(s)|$ *taken over* $R(T, T+H)$ *does not exceed* $\operatorname{Exp}\operatorname{Exp}((10A)^{-4}H_0)$

Remark 1

Theorem 5 is more fundamental. Given better kernels we can deduce better Theorems than Theorem 7.

Remark 2

We can state many corollaries to Theorem 7. We mention some of them here. The first is that $\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k} dt$ is $\gg (\log H)^{k^2}$ for all integers $k > 0$. The second is the validity of the same result for all irrational $k > 0$ on the assumption of Riemann hypothesis. In both these cases we need the condition $T > H \gg \log\log T$, and the constant implied by $\gg$ is independent of T and H and is supposed to be large enough The next is

$$\max_{T < t < T+H} |\zeta(\tfrac{1}{2}+it)| >$$

$$\left(\sum_{n < \frac{H}{100}} (d_k(n))^2 n^{-1} (C \log\log H)^{-1} \right)^{\frac{1}{2k}},$$

where $k > 1$ is any integer subject to $3^k < H$ and $C (> 0)$ is independent of k, T and H. (We still need the condition

$T > H \gg \log\log T$). By an optimal choice of k the RHS here was shown by R. Balasubramanian (Ref. On the frequency of Titchmarsh phenomenon for $\zeta(s)$—IV, J. of Number theory, to appear.) to exceed

$\operatorname{Exp}\left(\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{\frac{1}{2}}\right)$. It also gives similar results for $\max|\zeta(1+it)|$ and $\max|\zeta(\sigma+it)|$ where $\frac{1}{2} < \sigma < 1$. The results for $\max|\zeta(\frac{1}{2}+it)|$ and $\max|\zeta(1+it)|$ have some perfection which is not yet available for $\frac{1}{2} < \sigma < 1$. We expect $\max|\zeta(\sigma+it)| > \operatorname{Exp}\left(C_1\frac{(\log H)^{1-\sigma}}{(\log\log H)^{\sigma}}\right)$, $C_1 > 0$ a constant independent of T and H, but we have such a result with $\log\log H$ in place of $(\log\log H)^{\sigma}$.

We now deduce Theorem 5 from Theorem 6. As remarked already we can suppose that H exceeds a large constant depending on A. Suppose Theorem 5 is false. Then by the fact that the mean-value of the modulus of an analytic function over a disc is not less than its value at the centre, we see that the maximum of $|F(s)|$ taken over ($\sigma \geq \frac{2}{\log H}$, $T+1 \leq t \leq T+H-1$) does not exceed $C_A H^{4A}$. (We can now consider the series $\sum_{n=1}^{\infty}\left(a_n\lambda_n^{-s}\operatorname{Exp}\left(-\frac{\lambda_n}{Y}\right)\right)$ with for example $Y = \operatorname{Exp}(\frac{1}{2}\log H\log\log\log H)$. Let $0<\sigma_1<\sigma_2$. It is not hard to show that the absolute value of this series at $s = s_2 = \sigma+it$ does not exceed

$$|F(s_2)| + \frac{1}{2\pi}\int_{\substack{\operatorname{Re} w = \sigma_1-\sigma_2,\\ |\operatorname{Im} w - t| \leq \frac{1}{2}H}} |F(s_2+w)\, Y^{\operatorname{Re}(w)}\,\Gamma(w+1)|\left|\frac{dw}{w}\right|,$$

plus a negligible quantity. We may use things like $\int \left| \frac{dw}{w} \right|$, the integral taken over $|\mathrm{Im}(w)| < 1$ is $O((\log|\mathrm{Re}(w)|)^{-1})$. Now by applying Theorem 6 to F (s) (or it's truncation), it follows that

$$\max_{\sigma > \frac{2}{\log H}} \left(\frac{1}{H-2} \int_{T+1}^{T+H-1} |F(\sigma+it)|^2 dt \right)$$

$$> e^{-40} C_A \sum_{n<x} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H} \right)$$

Hence Theorem 5 is proved with $e^{-40} C_A$ in place of C_A.

Next we deduce Theorem 7 from 5. For this purpose we need the following Theorem 3' from which Gabriel deduced his Theorem 3 stated in § 2.

Theorem 3' (R. M. Gabriel)

Let R *be the closed rectangle with vertices* $z_0, \overline{z_0}, -z_0, -\overline{z_0}$. *Let* F (z) *be continuous on the boundary of* R *and regular in the interior of* R. *Then*

$$\int_L |F(z)|^{q^*} |dz|$$

$$< \left\{ \int_{P_1} |F(z)|^{q^*} |dz| \right\}^{\frac{1}{2}} \left\{ \int_{P_2} |F(z)|^{q^*} |dz| \right\}^{\frac{1}{2}}$$

for any $q^* > 0$, *where* L *is the line segment from* $\frac{1}{2}(\overline{z_0} - z_0)$ *to* $\frac{1}{2}(z_0 - \overline{z_0})$, P_1 *consists of three line segments connecting* $\frac{1}{2}(\overline{z_0} - z_0)$, $\overline{z_0}$, z_0 *and* $\frac{1}{2}(z_0 - \overline{z_0})$; *and* P_2 *is the mirror image of* P_1 *in* L.

Let σ_0 be the abscissa at which the maximum is attained.

Now $0 < \sigma_0 < A + 2$ since for $\sigma > A + 2$, we have,

$$\frac{1}{H} \int_T^{T+H} |F(\sigma + it)|^2 \, dt$$

$$= \sum_{n=1}^{\infty} |a_n|^2 \left(1 + O\left(\frac{n}{H}\right)\right) n^{-2\sigma},$$

where the O-constant is absolute. We now choose $F(z)$ in Theorem 3' to be $G(s)$ defined by,

$$G(s) = F\left(s + \sigma_0 + \frac{iH}{2}\right) \operatorname{Exp}\left(\left(\sin\left(\frac{w - s - \sigma_0 - \frac{1}{2} iH}{2A + 4}\right)\right)^2\right),$$

where we have written s for z. We now choose $z_0 = s_0 = \sigma_0 + \frac{iH}{4}$, and R, L, P_1, P_2 accordingly as in Theorem 3'. We write $w = iv$ and average the inequality provided by Theorem 3' with respect to v from $v = T - \frac{1}{2} H$ to $T + \frac{3H}{2}$. Theorem 7 follows since the horizontal sides in P_1 and P_2 contribute a negligible quantity by our assumption on the size of $|F(s)|$. The details are as follows. We have,

$$\frac{1}{2H} \int_W \int_L |G(s)| \, dv \, dt < \left(\frac{1}{2H} \int_W \int_{\bar{P}_1} |G(s)| \, dv dt\right)^{\frac{1}{2}}$$

$$\left(\frac{1}{2H} \int_W \int_{\bar{P}_2} |G(s)| \, dv dt\right)^{\frac{1}{2}} + \text{a negligible quantity},$$

where $\bar{P}_1$ and $\bar{P}_2$ denote the vertical portions of P_1 and P_2. Our choice of σ_0 poses a slight difficulty. To avoid this we choose σ_0 such that LHS here is maximum. This proves Theorem 7, since the main term on the RHS is now,

$$< \left(\frac{1}{2H} \int_{W} \int_{\overline{P}_2} | (G(s) | \, dv \, dt \right)^{\frac{1}{2}}$$

$$\left(\frac{1}{2H} \int_{W} \int_{L} | (G(s) | \, dv \, dt \right)^{\frac{1}{2}}$$

Remark

The result

$$\frac{1}{T} \int_{T}^{2T} | \zeta(\tfrac{1}{2}+it) |^{2k} \, dt > C_k (\log T)^{k^2},$$

for irrational $k > 0$ seems to be difficult. However using the theorem that every real irrational is the sum of two real numbers each of which have partial quotients < 4 in their simple continued fraction expansions one may hope to get some day the lower bound $C_k (\log T)^{k^2} (\log \log T)^{-\frac{1}{2} k^2}$.

The theorem on the decomposition into two real numbers referred to here is due to Marshal Hall Jr. (Ref. Annals of Maths. 48 (1947), 966-993). For a simple proof of a weaker result see J. W. S. Cassels, Mathematika 3 (1956), 109-110.

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