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A NOTE TO A PAPER BY RAMACHANDRA ON TRANSCENDENTAL NUMBERS

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TO

PROFESSOR Th. SCHNEIDER ON HIS SEVENTIETH BIRTHDAY

§ 1 Introduction

A typical result proved is that out of the numbers 2^{π^n} $(n = 1, 2, \ldots, N)$ the number of algebraic numbers does not exceed $\frac{1}{2} + \sqrt{(4N - 4 + \frac{1}{4})}$ The same bound is true for the numbers 2^{t^n} where t is any transcendental number and more generally of the numbers $\mathrm{Exp}\,(\alpha t^n)$ where α is any non-zero complex number. Next if $\wp(z)$ is any Weierstrass elliptic function with algebraic invariants g_2, g_3 the corresponding bound for the numbers $\wp(\alpha t^n)$ is $\frac{1}{2} + \sqrt{(8N - 8 + \frac{1}{4})}$ (Here the value ∞ of $\wp(z)$ is regarded as algebraic by convention). Actually our argument in § 2 shows that the number of algebraic numbers in 2^{π^n} $(n = M+1, M+2, \ldots M+N)$ does not exceed $\frac{1}{2} + \sqrt{(4N - 4 + \frac{1}{4})}$ and similar results in other cases stated above. We show that these results are easy corollaries to the following Theorem (see [2]).

Theorem

(1) Let a *and* b *be non-zero complex numbers such that* $\frac{a}{b}$ *is irrational. Let* $\alpha_1, \alpha_2, \alpha_3$ *be complex numbers such that* $t_1 \alpha_1 + t_2 \alpha_2 + t_3 \alpha_3 = 0$ (t_1, t_2, t_3 *integers*) *is possible only when* $t_1 = t_2 = t_3 = 0$. *Then at least one of the six numbers* $\operatorname{Exp}(a\alpha_i)$, $\operatorname{Exp}(b\alpha_i)$ $(i = 1, 2, 3)$ *is transcendental.*

(2) *Let* a *and* b *be non-zero complex numbers such that* $\frac{a}{b}$ *is irrational. Let* $\alpha_1, \alpha_2, \ldots, \alpha_5$ *be complex numbers such that* $t_1 \alpha_1 + t_2 \alpha_2 + \cdots + t_5 \alpha_5 = 0$ ($t_1, \ldots, t_5$ integers) is possible only when $t_1 = t_2 = \ldots = t_5 = 0$. *Then at least one of the ten numbers* $\wp(a\alpha_i)$, $\wp(b\alpha_i)$, $(i = 1 \text{ to } 5)$ *is transcendental.*

Remarks

The first part of the Theorem is originally due to Siegel, Schneider, and Gelfond (see Problem 1 on page 138 of [4]). The first part was rediscovered and the second part was proved by K. Ramachandra in his paper [2], by developing the method of Siegel, Schneider, and Gelfond. Some progress in the direction of the Theorem was made (around the same time) independently by S. Lang, who in particular rediscovered the first part of the Theorem (see page 119 of A. Baker's book [1]).

§ 2 Proof of the Results

We first prove the following Lemma.

Lemma

Let S *be a non-empty set of natural numbers and* S(N, M) *the sub-set of those natural numbers contained in* [M, M + N − 1], (*where* M *and* N *are any two natural numbers*) *Suppose that* *for every non-zero integer* r, *the equation*

$$x + r = y, \ (x, y \text{ in } S),$$

does not have more than D *solutions(where D is a natural number independent of r). Let* f(N, M) *be the number of natural numbers in* S(N, M). *Then there holds,*

$$f(N, M) \leq \tfrac{1}{2} + \sqrt{(2DN - 2D + \tfrac{1}{4})}$$

Proof

Let g (r, N, M) denote the number of solutions of the equation in the Lemma with the restriction that x, y should be in S (N, M). Then

$$\begin{aligned}(f(N, M))^2 &= \sum_{-(N-1) \leq r \leq N-1} g(r, N, M) \\ &= \sum_{r=0} + \sum_{r \neq 0} \\ &\leq f(N, M) + 2D(N-1).\end{aligned}$$

This proves the Lemma completely.

Using the lemma our results can be deduced easily from the Theorem as follows. Take S to be the set of those natural numbers for which Exp (αt^n) is algebraic. We take M = 1. In the first part of the Theorem we take $a = \alpha$, $b = \alpha t^r$. Then the Theorem tells us that we can take D = 2. For our assertions about $\wp(z)$ we can take $a = \alpha$, $b = \alpha t^r$. Then the Theorem tells us that we take D = 4.

This proves our assertions completely.

Remark

In a paper to appear [3], R. Balasubramanian and K. Ramachandra prove that the number of algebraic numbers amongst $2^{\pi}, 2^{\pi^2}, \ldots, 2^{\pi^N}$ is $\sqrt{2N}\,(1+o(1))$. They also prove similar improvements of results in section 2.

§ 3. **Further Results.** The Gelfond-Siegel-Schneider method (of proving the transcendence of e^{π} and $2^{\sqrt{2}}$) was developed in a deep way by Gelfond to prove the algebraic independence of 2^{β} and 2^{β^2} where $\beta = \sqrt[3]{2}$. These researches of Gelfond have been continued by R. Tijdeman, D. Brownawell, and M. Waldschmidt. More profound results have been obtained by G. V. Choodnovski and we now quote a result from his paper: Algebraic Independance of values of exponential and elliptic functions (Proceedings of the International Congress of Mathematicians, Helsinki (1978) pages 339 - 350). *Let* $\alpha_1, \ldots, \alpha_M$ *be complex numbers linearly independent over the rationals and let* $\beta_1, \beta_2, \ldots, \beta_N$ *be complex numbers linearly independent over the rationals where* $MN \geq (n+1)(M+N)$, M, N, *and* n *being fixed natural numbers. Then at least* $n+1$ *of the* MN *numbers* $\mathrm{Exp}(\alpha_i \beta_j)$ *are algebraically independent.*

Let t be any fixed transcendental number and let $t_1, \ldots, t_n$ be any n fixed algebraically independent complex numbers. Let α be a fixed non-zero complex number. Putting $\alpha_i = \alpha t^{r_i}$, $\beta_i = \alpha t^{m_j}$, where r_i $(i = 1$ to $M)$ are any non-zero distinct integers and m_j $(j = 1$ to $N)$ are distinct natural numbers, we deduce the following corollary. Let S be the set of those natural numbers m for which $\mathrm{Exp}(\alpha t^m)$ depends algebraically on $t_1, \ldots, t_n$ and further let S_0 be the set of those numbers in S which satisfy $x \leq m \leq x + y$, where $x > 1$, $y > 1$. We now set $M = n+2$ and by choosing N to be a large constant and m_j to be in S_0 we deduce that one at least of the numbers $\mathrm{Exp}(\alpha t^{r_i + m_j})$ does not belong to S_0. From this we deduce that the number of integers in S_0 does not exceed Cy^{θ} where $\theta = 1 - \frac{1}{n+3}$, and C depends only on n. The last deduction is facilitated by the following lemma which is an extension of the lemma proved in section 2.

Extension

Let k *be a natural number and* S_0 *a subset of natural numbers consisting of at least two and at most finitely natural numbers. Consider the difference set* R *consisting of all non-zero differences of numbers in* S_0. *Let* T *be any non-empty subset of* R^k *the set of all possible* k*-tuples of numbers in R. For any integer* r *in* R, *put* $S_r = \{ a \mid a \text{ in } S_0,\ a + r \text{ in } S_0 \}$. *Now for* $r = (r_1, \dots, r_k)$ *in* R^k *put* $S_r = S_{r_1} \cap \dots \cap S_{r_k}$.

Then

$$\sum_{r \text{ in } T} \sum_{a \text{ in } S_r} 1 = \sum_{a \text{ in } S_0} \sum_{r \text{ in } T,\ S_r \text{ containing } a} 1$$ *and*

the L. H. S. *is not more than* $(\max_{r \text{ in } T} \sum_{a \text{ in } S_r} 1)\ (\sum_{r \text{ in } T} 1)$.

Further if T *consists of all possible* $r = (r_1, \dots, r_k)$, *where* $r_1, \dots, r_k$ *are distinct then* R. H. S. *here is* $\gg$ *and* $\ll$ $(\sum_{a \text{ in } S_0} 1)^{k+1}$ *where the implied constants depend only on* k.

Remark

Taking $k = n + 2$ and S_0 as described before the extension we get the result stated since $\max_{r \text{ in } T} \sum_{a \text{ in } S_r} 1$ is bounded by a constant depending only on n by the deep result of Choodnovski.

Proof of the extension follows by interchanging the summation. The bounds for the R. H. S. comes from the fact that $\sum_{r \text{ in } T,\ S_r \text{ containing } a} 1$ is closely related to the number of combinations of $(\sum_{a \text{ in } S_0} 1)$, taken k at a time.

We now state the final result which we have deduced as an easy corollary to the deep result of G. V. Choodnovski

Final Result

Let $\alpha, t_1, \ldots, t_n$ *be* $n+1$ *non-zero complex numbers where* n *is a fixed natural number. Let* m *run through those natural numbers for which* $\operatorname{Exp}(\alpha t^m)$ *(*t *being a fixed transcendental number) depends algebraically on* $t_1, \ldots, t_n$. *Put* $N(x) = \sum_{m \leq x} 1$. *Then* $N(x+y) - N(x)$ *does not exceed* $C y^{\theta}$ *where* $\theta = 1 - \frac{1}{n+3}$ *and* C *is an effective positive constant depending only on* n. *Here as usual* $x > 1$, *and* $y > 1$.

Remark 1

When $t_1, \ldots, t_n$ are algebraic numbers the results of section 2 are better.

Remark 2

This note began with the observation that in section 2 the set S does not contain any sub-set with D+1 elements in arithmetical progression (this follows immediately from the Theorem) and so by Szemeredi's Theorem $f(N, 1) = o(N)$. Next in a correspondence with us Professor R. Tijdeman pointed out that it is possible to use Roth's theorem in the direction of Szemeredi's Theorem and get

$$f(N, 1) = O\left(\frac{N}{\log\log N}\right).$$

Remark 3

It is possible to reduce θ to $1 - \frac{1}{n+2}$ in the final result.

Remark 4

The lemma can be improved slightly. In particular the number of algebraic numbers amongst $2^{\pi}, 2^{\pi^2}, \ldots, 2^{\pi^N}$ is $\leq \frac{1}{2}(-1 + \sqrt{(16N-7)})$.

Remark 5.

Let $\phi(N)$ be the number of transcendental numbers of the form $2^{t^{f(n)}}$ where $1 \leq n \leq N$ and t is a fixed transcendental. If $f(n) = n^2$ then $\phi(N) = \left(1 + O\left(\sqrt{\left(\frac{\log\log N}{\log N}\right)}\right)\right)N$. This is a consequence of the fact that the number of integers in $(-N^2, N^2)$ which have at most two representations of the form $f(n_1) - f(n_2)$ $(n_1 \neq n_2)$ is $O\left(N^2 \frac{\log\log N}{\log N}\right)$. Next since there is a positive integer which is a difference of 2 positive cubes in 3 different ways it follows that if $f(n) = n^3$, $\phi(N) \gg N$ for $N > 10^{30}$.

Starting from $37 = 4^3 - 3^3 = \left(\frac{10}{3}\right)^3 - \left(\frac{1}{3}\right)^3$ we get by the chord process a new point. The tangent process at this point gives $(19(19^3 + 2.18^3))^3 - (18(2.19^3 + 18^3))^3$ $= (7(19^3 - 18^3))^3\, 37$.

References

1. A. Baker, Transcendental Number Theory, Cambridge University Press (1975).

2. K. Ramachandra, Contributions to the theory of transcendental numbers — I, Acta Arith., 14 (1968) 65 — 72, — II ibid., 14 (1968) 73 — 88.

3. R. Balasubramanian and K. Ramachandra, Transcendental number Theory and a lemma in combinatorics (to appear)

4. Th. Schenider, Einfubruug in die Transzendenten Zahlen, Springer - Verlag (1957).

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