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► **To cite this version:**

A. Venugopalan. Primes between  $p_{-n} + 1$  and  $p_{-n} + 1^2 - 1$ . Hardy-Ramanujan Journal, 1983, Volume 6 - 1983, pp.45 - 48. 10.46298/hrj.1983.99 . hal-01104264

**HAL Id: hal-01104264**

**<https://hal.science/hal-01104264>**

Submitted on 16 Jan 2015

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# PRIMES BETWEEN $p_n + 1$ AND $p_{n+1}^2 - 1$

By

A. VENUGOPALAN

## § 1. Introduction

We prove the following four theorems. We begin with some notation. Let  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  be the sequence of all prime numbers. Let  $Q$  be the product of the first  $n$  prime numbers. Let  $Q_i = Q/p_i^{-1}$  for  $i = 1, 2, 3, \dots, n$ .

Let  $K = n$ . Let  $J$  stand for  $\sum_{i=1}^n a_i Q_i - bQ$

### Theorem 1

We have,

$$a_1 = 1 \quad \dots \quad a_n = 1 \quad 0 < b < K \quad x^J = x + \frac{1}{x} + f(x) + f\left(\frac{1}{x}\right)$$

where  $f(x) = \sum_{\substack{m \\ p_{n+1}^2 + 1 < m < KQ, (m, Q) = 1 \\ + \sum_{\substack{m \\ p_n + 1 < m < p_{n+1}^2 - 1, (m, Q) = 1}} x^m$

and \* denotes the omission of some integers  $m$ .

### Theorem 2

$$a_1 = 1 \quad \dots \quad a_n = 1 \quad 0 < b < K \quad x^{J^2} = 2x + 2\phi(x)$$

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where  $\phi(x) = \sum_{x \in \mathbb{N}} x^2$

$$p_{n+1}^2 \sqrt{1} < m < KQ, (m, Q) = 1$$

$$+ \sum x^2$$

$$p_{n+1}^2 + 1 < m < p_{n+1}^2 - 1, (m, Q) = 1$$

and  $\bullet$  denotes the omission of some integers  $m$ .

**Theorem 3**

Let  $(a'_1, \dots, a'_n)$  be the unique solution of

$$-2 \equiv \sum_{i=1}^n a'_i Q_i \pmod{Q}, \text{ with}$$

$$0 < a'_i < p_i - 1, (i = 1, 2, \dots, n).$$

Then

$$p_1-1 \quad p_2-1 \quad \dots \quad p_n-1$$

$$a_1 = 1 \quad a_2 = 1 \quad \dots \quad a_n = 1 \quad 0 < b < K$$

$$a_1 \neq a'_1 \quad a_2 \neq a'_2 \quad a_n \neq a'_n$$

$$= \sum_{x \in \mathbb{N}} x^m \pmod{(m(m+2), Q)} = 1$$

where the sum on the right sums over the relevant range for  $m$ .

**Theorem 4**

We have,

$$\sum_{a_1} \dots \sum_{a_n} \sum_b x^{j^2} = \sum_{x \in \mathbb{N}} x^{m^2} \pmod{(m(m+2), Q)} = 1$$

The sum over  $m$  on the right bring over the same set of integers as in Theorem 3.

**Remark 1**

Note that  $a'_1 = 0$  and that  $1 < a'_i < p_i - 1$   
 $(i=2, 3, \dots, n)$

**Remark 2**

In the first two theorems the  $m$ 's that satisfy

$p_n + 1 < m < p_{n+1}^2 - 1$  are precisely *all the primes in this interval*. In the next two theorems they are all the twin primes in this interval.

§ 2 Proof

The proofs of theorems 1 and 2 follow from the following two remarks.

First given any integer  $c$  there is a unique solution of

$$\sum_{i=1}^n a_i Q_i \equiv c \pmod{Q} \text{ subject to } 0 < a_i < p_i - 1$$

( $i = 1, 2, \dots, n$ ). Out of these solutions  $(c, Q) = 1$  is satisfied if and only if  $1 < a_i < p_i - 1$  ( $i = 1, 2, 3, \dots, n$ ).

The proof of theorems 3 and 4 follow from the following remark.

Subject to  $1 < a_i < p_i - 1$  for all  $i$  we have already secured  $(m, Q) = 1$ . If in addition  $m + 2$  is to be coprime to  $Q$  we should have

$$(\sum a_i Q_i - \sum a'_i Q_i, Q) = 1, \text{ i. e. } a_i - a'_i \neq 0 \text{ for each } i.$$

§ 3. Further Remarks

We can find by the method above conditions to ensure  $(m(m+2)(m+6), Q) = 1$  and so on. Next one can easily get a formula for the  $n^{\text{th}}$  prime from Theorem 2. It is :

$$p_{n+1}^2 - 1 = \left[ -\log \left( \frac{1}{2} \sum_{a_1=1}^{p_1-1} \sum_{a_2=1}^{p_2-1} \dots \sum_{a_n=1}^{p_n-1} \sum_{b=1}^n e^{-(\sum a_i Q_i - bQ)^2 - \frac{1}{c}} \right) \right]$$

What we have done corresponds nearly to the Eratosthenese sieve. It will be interesting to modify our investigations in a way which correspond to Brun's sieve.

#### **Acknowledgement**

I am thankful to Professor K Ramachandra for encouragement.

#### **Reference**

- 1) K. Ramachandra, Viggo Brun (13—10—1885 to 15—8—1978), *The Mathematics Student*, Vol. 49, No. 1 (1981) p 87—95

Manuscript completed on 22 September 1981.

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