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PRIMES BETWEEN $p_n + 1$ AND $p_{n+1}^2 - 1$

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§ 1. Introduction

We prove the following four theorems. We begin with some notation. Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of all prime numbers. Let Q be the product of the first n prime numbers. Let $Q_i = Qp_i^{-1}$ for $i = 1, 2, 3, \ldots, n$.

Let K = n. Let J stand for $\sum_{i=1}^{n} a_i Q_i - bQ$

Theorem 1

We have,

$$\sum_{a_1=1}^{p_1-1} \cdots \sum_{a_n=1}^{p_n-1} \sum_{0\leq b\leq K} x^J = x + \frac{1}{x} + f(x) + f\left(\frac{1}{x}\right)$$

where $f(x) = \sum\nolimits^{*}_{p_{n+1}^2 + 1 \leq m \leq KQ,\ (m,Q)=1} x^m$

$$+ \sum_{p_n + 1 \leq m \leq p_{n+1}^2 - 1,\ (m,Q)=1} x^m$$

and * denotes the omission of some integers m.

Theorem 2

$$\sum_{a_i=1}^{p_1-1} \cdots \sum_{a_n=1}^{p_n-1} \sum_{0\leq b\leq K} x^{J^2} = 2x + 2\phi(x)$$

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where $$\phi(x) = \sum^{*}_{\substack{p_{n+1}^{2}+1 \leqslant m \leqslant KQ,\ (m,\,Q)=1}} x^{m^2} + \sum_{\substack{p_{n}^{2}+1 \leqslant m \leqslant p_{n+1}^{2}-1,\ (m,\,Q)=1}} x^{m^2}$$

and $*$ *denotes the omission of some integers* m.

Theorem 3

Let $(a'_1, \ldots, a'_n)$ *be the unique solution of*

$$-2 \equiv \sum_{i=1}^{n} a'_i Q_i \bmod Q, \textit{ with}$$
$$0 < a'_i < p_i - 1, \ (i = 1, 2, \ldots, n).$$

Then

$$\sum_{\substack{a_1=1 \\ a_1 \neq a'_1}}^{p_1-1} \sum_{\substack{a_2=1 \\ a_2 \neq a'_2}}^{p_2-1} \cdots \sum_{\substack{a_n=1 \\ a_n \neq a'_n}}^{p_n-1} \sum_{0 \leqslant b < K} x^{J} = \sum_{(m(m+2),\,Q)=1} x^{m}$$

where the sum on the right sums over the relevant range for m.

Theorem 4

We have,

$$\sum_{a_1} \cdots \sum_{a_n} \sum_{b} x^{J^2} = \sum_{(m\,(m+2),\,Q)=1} x^{m^2}$$

The sum over m on the right bring over the same set of integers as in Theorem 3.

Remark 1

Note that $a'_1 = 0$ and that $1 \leqslant a'_i \leqslant p_i - 1$
$$(i = 2, 3, \ldots, n)$$

Remark 2

In the first two theorems the m's that satisfy

$p_n + 1 < m < p_{n+1}^2 - 1$ are precisely *all the primes in this interval*. In the next two theorems they are all the twin primes in this interval.

§ 2 Proof

The proofs of theorems 1 and 2 follow from the following two remarks.

First given any integer c there is a unique solution of

$$\sum_{i=1}^{n} a_i Q_i \equiv c \pmod Q \text{ subject to } 0 \leqslant a_i \leqslant p_i - 1$$

(i = 1, 2, ..., n). Out of these solutions (c, Q) = 1 is satisfied if and only if $1 < a_i < p_i - 1$ (i = 1, 2, 3, ..., n).

The proof of theorems 3 and 4 follow from the following remark.

Subject to $1 < a_i < p_i - 1$ for all i we have already secured (m, Q) = 1. If in addition m + 2 is to be coprime to Q we should have

$(\Sigma a_i Q_i - \Sigma a'_i Q_i , Q) = 1$, i. e. $a_i - a'_i \neq 0$ for each i.

§ 3. Further Remarks

We can find by the method above conditions to ensure (m (m+2) (m+6), Q) = 1 and so on. Next one can easily get a formula for the n^{th} prime from Theorem 2. It is ;

$$p_{n+1}^2 - 1 = \left[-\log\left(\frac{1}{2} \sum_{a_1=1}^{p_1-1} \sum_{a_2=1}^{p_2-2} \cdots \right.\right.$$

$$\left.\left. \cdots \sum_{a_n=1}^{p_n-1} \sum_{b=1}^{n} e^{-(\Sigma a_i Q_i - bQ)^2} - \frac{1}{e} \right) \right]$$

What we have done corresponds nearly to the Eratosthanese sieve. It will be interesting to modify our investigations in a way which correspond to Brun's sieve.

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Reference

1) K. Ramachandra, Viggo Brun (13—10—1885 to 15—8—1978), The Mathematics Student, Vol. 49, No. 1 (1981) p 87—95

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