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ANNOUNCEMENTS

DISTINGUISHED AWARDS OF THE HARDY-RAMANUJAN SOCIETY have been conferred on J. PINTZ, H. IWANIEC, C.J. MOZZOCHI, H. MAIER, C. POMERANCE, C. SPIRO, E. WIRSING, H.L. MONTGOMERY, R.C. VAUGHAN, M.N. HUXLEY and N. WATT for their discoveries of which the following are quite famous.

THEOREM. (H. IWANIEC and J. PINTZ). *For all $n \geq n_0(\epsilon)$ there holds*

$$p_{n+1} - p_n < p_n^{1/2+1/21+\epsilon}$$

where p_n is the nth prime number and $0 < \epsilon < 1$ is an arbitrary constant.

THEOREM. (C.J. MOZZOCHI). *The above theorem is true with $\frac{1}{2} + \frac{1}{21} - \delta$ (in place of $\frac{1}{2} + \frac{1}{21}$) where $\delta = \frac{1}{4480}$.*

REMARK. This result is superseded by the result [45] with $\frac{1}{2} + \frac{1}{22}$ in place of $\frac{1}{2} + \frac{1}{21}$ due to S.-t LOU and Q. YAO.

THEOREM. (H. IWANIEC and C.J. MOZZOCHI). *Let $d(n)$ denote the number of positive integers which divide n. Let (for $x \geq 1$)*

$$E_1(x) = \sum_{1 \leq n \leq x} d(n) - x\log x - (2\gamma - 1)x$$

where γ is the Euler's constant. Let $R(x)$ (for $x \geq 1$) denote the number of lattice points (i.e. points with integral co-ordinates) (m, n) with $m^2 + n^2 \leq x$. Let

$$E_2(x) = R(x) - \pi x.$$

Then for all ϵ with $0 < \epsilon < 1$ and all $x \geq x_0(\epsilon)$ we have

$$| E_1(x) | + | E_2(x) | < x^{7/22+\epsilon}.$$

THEOREM (H. MAIER and C. POMERANCE). *Let p_n denote the nth prime number. Then*

$$\lim_{n\to\infty} Sup(p_{n+1} - p_n)\{\log p_n \frac{\log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}\}^{-1} \geq C$$

where $C = \eta e^{\gamma}$ *where* η *is the solution of* $\frac{4}{\eta} - e^{-4/\eta} = 3(\eta = 1.31256,...)$.

THEOREM (H. MAIER). *We have*

$$\lim_{h \to \infty} Inf \quad (p_{n+1} - p_n)(\log p_n)^{-1} \leq 0.248...$$

where p_n *denotes the nth prime number.*

REMARK. $0.248... = De^{-\gamma}$ *where* D *is the constant corresponding to* $0.248...$ *in his first paper by M.N. HUXLEY and* γ *is the Euler's constant.*

THEOREM (C. SPIRO). *Let* $d(n)$ *denote the number of positive integers which divide* n. *Then for infinitely many positive integers* n *there hold* $d(n) = d(n + 5040)$.

REMARK. D.R. HEATH-BROWN has proved the same for the equation $d(n) = d(n + 1)$.

THEOREM. (E. WIRSING). *Let* α *be an algebraic number of degree* g *and* ζ *an algebraic number of degree* h. *Let* $0 < \varepsilon < 1$ *be arbitrary. Then there exists a constant* $C = C(\alpha, h, \varepsilon)$ *depending only on* α, h *and* ε *such that for all* $\zeta \neq \alpha$ *there holds with* $\mu = 2h$

$$| \alpha - \zeta | > C(H(\zeta))^{-\mu-\varepsilon}$$

where $H(\zeta)$ *is the maximum of the absolute values of the coefficients of the irreducible polynomial with integer coefficients of which* ζ *is a root.*

REMARK. The earlier record was $\mu = min_{m \geq 1}\{mh^{m-1}(\frac{g}{m!})^{1/m}\}$ due to K. RAMACHANDRA and A. BAKER independent of each other. WIRSING's record was superseded by that of W.M. SCHMIDT who obtained the result with $\mu = h + 1$.

THEOREM.(E. WIRSING). *There exists a subset* **P** *of the set of all prime numbers with the following properties:*

(1) *The number of primes in* **P** *which are* $\leq x$ *is* $0((x \log x)^{1/3})$ *and*

(2) *there exists a positive constant* n_0 *such that for all integers* $n \geq n_0$ *there holds*

$$2n + 1 = p_1 + p_2 + p_3$$

where p_1, p_2, p_3 *belong to* **P**.

REMARK. WIRSING also proves a corresponding result for expressing a given integer as sums of a thin set of squares. The theorem mentioned above is based upon I.M. VINOGRADOV's famous result on the number of representations of a large odd number as sum of three primes and some considerations of probability theory and stochastic processes originated by P. ERDÖS and M.B. NATHANSON.

There is an old conjecture due to GOLDBACH (made in a letter to L. EULER) which states that every even number ≥ 4 can be expressed as a sum of two suitably chosen prime numbers. Regarding this conjecture the following theorem is a splendid achievement.

THEOREM (H.L. MONTGOMERY and R.C. VAUGHAN). *Let $E(x)$ denote the number of even numbers $\leq x$ for which the conjecture is not true. Then*

$$E(x) < x^{\theta}$$

where θ is a number < 1 and independent of x.

REMARK. The method of H.L. MONTGOMERY and R.C. VAUGHAN allows one to find out a specific constant θ and the latest score is the bound $Cx^{\theta}(\theta = \frac{24}{25})$ with a numerical constant $C > 0$, due to CHEN-JING-RUN and PAN-CHENG-DONG. However the first attempt in this direction is due to G.H. HARDY and J.E. LITTLEWOOD who showed on G.R.H. that one can take any $\theta > \frac{1}{2}$. The method employed is a development of the circle method.

THEOREM (H.L. MONTGOMERY and R.C. VAUGHAN). *Let n be a positive integer and let $a_1 = 1 < a_2 < \dots < a_{\varphi(n)}$ be all the integers not exceeding n and prime to it. Then*

$$\frac{1}{n}\sum_{i=1}^{\varphi(n)-1} (a_{i+1} - a_i)^2 < C(\frac{n}{\varphi(n)}),$$

where C is a numerical constant independent of n.

REMARK. The earlier record was due to C. HOOLEY who proved that for every α satisfying $1 < \alpha < 2$ we have

$$\frac{1}{n}\sum_{i=1}^{\varphi(n)-1} (a_{i+1} - a_i)^{\alpha} < C(\alpha)(\frac{n}{\varphi(n)})^{\alpha-1}$$

where $C(\alpha) > 0$ depends only on α. (The result in the theorem was conjectured by P. ERDÖS). In fact H.L. MONTGOMERY and R.C. VAUGHAN proved this for every fixed $\alpha > 1$.

THEOREM (E. BOMBIERI and H. IWANIEC). *Let*

$$\zeta(s) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{du}{u^s}\right) + \frac{1}{s-1}.$$

Then for every $\varepsilon(0 < \varepsilon < 1)$ *there holds for all* $t \geq t_0(\varepsilon)$,

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| < t^{\frac{9}{56}+\varepsilon}$$

THEOREM (M.N. HUXLEY and N. WATT). *With the same notation, we have, for all* $t \geq 10$

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| < Ct^{\frac{9}{56}-\frac{1}{560}}(\log t)^{\frac{47}{32}}$$

where C *is a numerical constant* (independent of t).

The distinguished awards carry 20 Pounds (or 40 Dollars) each, meant for the purchase of the collected work (one volume) of mathematicians such as HARDY, LITTLEWOOD, DAVENPORT, TURAN, and LINNIK. The magnitude of the award is not important. The award is an indication of the respect of the Society to their work. Mathematicians like E. BOMBIERI and W.M. SCHMIDT whose names figure above certainly deserve the award but they have been recognized and rewarded by other awards and hence we do not pay them the amount.

R. SITARAMACHANDRARAO of WALTAIR, INDIA has been awarded a prize of Rs.100 for his work

THEOREM (R.SITARAMACHANDRARAO). *We have,*

$$\int_0^1 \left|\zeta\left(\frac{1}{2} + it, \alpha\right)\right|^2 d\alpha = \log\left(\frac{t}{2\pi}\right) + \gamma + 0\left((\log t)^{\frac{3}{8}} t^{\frac{-3}{16}}\right)$$

where $t \geq 2, \gamma$ *is the Euler's constant and*

$$\zeta(s, \alpha) = \sum_{n=1}^{\infty}\left(\frac{1}{(n+\alpha)^s} - \int_n^{n+1}\frac{du}{(u+\alpha)^s}\right) + \frac{(1+\alpha)^{1-s}}{s-1}.$$

As stated already the magnitude of the award is not important but is an indication of the respect of the Society to the work and dedication of his service to Indian analytic number theory.