My memories of Srinivasan

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It was probably in June of 1981 that I first met S. Srinivasan. We shared an office at the Tata Institute of Fundamental Research (TIFR) located in Colaba, a suburb of Mumbai, India. I was a post-doctoral fellow there at that time. Even though my appointment started officially on July 1, 1981, I came a month earlier to attend a conference organized by Krishnaswami Alladi, in Mysore, India from June 3-6. The conference proceedings were published as Volume 938 of Springer's Lecture Notes in Mathematics.

In the main building of the Tata Institute, I was assigned an office, sharing it with Srinivasan who was already there. In those days, cigarette smoking was allowed inside all buildings including sadly the cafeterias (thankfully, it is no longer the case). I detested this practice and abhorred the smell and the stuffy air that went with it. Srinivasan was aware of my views on the matter and never smoked in our office and tried as far as possible to prevent me seeing him with a cigarette in his hand. I respected him for this courtesy.

In those halcyon days, Srinivasan was not married and he would take all his meals in the TIFR canteens just like I did, and so we would invariably eat together at lunch and dinner, often in the east canteen, which had a more authentic Indian cuisine. But we never had breakfast together because Srinivasan was a late riser and I was an early bird. So, I would usually eat alone at 8am when the west canteen opened for breakfast. The east canteen didn't serve breakfast.

Srinivasan would come to the office around 10am and around 10.30am we would both go for morning coffee served in the west canteen. We usually sat at a table near the large floor to ceiling window which commanded a spectacular view of the lush TIFR gardens and the turbulent waves of the Arabian Sea cascading against the craggy breakwaters. After our morning coffee, we would go and check for mail in the mailroom. Even though the mail would be delivered to our rooms by lunch time, this was a daily ritual with Srinivasan. After this, we would go back to our office and work until lunch time which was around 1pm.

That year, I was giving a seminar course called "The Analytic Theory of Modular Forms" which met once a week. Daily, I was busy preparing material for this course. Later, I, along with Subhashis Nag, were recruited to give two graduate courses, me in number theory and Nag in complex analysis. There were only four "official" students, but the class had some "unofficial participants" including Dipendra Prasad, Dinesh Thakur, Dilip Patil and sometimes Srinivasan. I taught a three semester course consisting of analytic number theory (based on Davenport's Multiplicative Number Theory), algebraic number number theory (based on Hecke's lectures) and modular forms (based on the last section of Serre's Course in Arithmetic). One of the problems that intrigued me in the seminar course was the question of analytic approaches to Ramanujan's celebrated conjecture about the τ-function. It is well-known that Deligne had solved this as a consequence of his work on the Weil conjectures. I decided to focus on the analytic approach in my seminar course, and Srinivasan was always there for every lecture. At first, I had quite a number of people in the room but as time marched forward, the audience dwindled. At one time it was only Srinivasan who was in attendance!

There was one particular paper I honed in on. This was Selberg's 1965 paper [Sel63] dealing with the relationship between Kloosterman sums and the Ramanujan conjecture. Perhaps a few mathematical words will help the reader understand the essence of this paper.

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As is well-known, the Ramanujan τ-function can be defined as the sequence of coefficients defined by an infinite product:

$$\sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

Ramanujan conjectured that $|\tau(p)| \leq 2p^{11/2}$ for every prime number p. Using the theory of Poincaré series, one can show that there is an absolute constant $A > 0$ such that for every $n > 1$,

$$\tau(n) = An^{11/2} \sum_{c=1}^{\infty} \frac{S(1,n,c)}{c} J_{11}\left(\frac{4\pi\sqrt{n}}{c}\right), \tag{0.1}$$

where $S(1,n,c)$ is the Kloosterman sum

$$\sum_{a (\operatorname{mod} c), (a,c)=1} e^{\frac{2\pi i}{c}(a+na')},$$

with $aa' \equiv 1 \pmod{c}$ and J_{11} is the Bessel function (which can be viewed as a continuous analogue of the Kloosterman sum) given by the integral

$$J_{11}(z) := \frac{1}{2\pi i} \int_C t^{-12} e^{z(t-1/t)/2} dt.$$

This formula (0.1) is in itself remarkable and one can only gaze it in wonder and awe.

In his terse paper, Selberg [Sel63] conjectured that if one had some nice estimate on averages of Kloosterman sums, then Ramanujan's conjecture follows. More precisely, he conjectured that

$$\sum_{c \leq x} \frac{S(1,n,c)}{c} = O(x^{\epsilon}),$$

for any $\epsilon > 0$. Here the implied constant is absolute. Linnik made a similar conjecture and since then, this has been called the Selberg-Linnik conjecture. Notice that Weil's estimate for the Kloosterman sum leads to $O(x^{1/2+\epsilon})$. So the prediction is that substantial cancellation is taking place in the summation.

Selberg offered no explanation of this implication and I struggled with this for more than a week before I asked Srinivasan for his thoughts. After a day, he came back with some insightful remarks regarding the Bessel function and how to estimate it cleverly in the Petersson formula. This finally led to a proof which I presented in the seminar course. The essential idea was to use the Bessel function identity

$$\frac{2kJ_k(x)}{x} = J_{k+1}(x) + J_{k-1}(x) \tag{0.2}$$

and since the sum on the right hand side of the formula (0.1) with J_9 instead of J_{11} is identically zero (because there are no cusp forms of weight 10), we can inject an induction argument using equation (0.2). This allows us to replace J_{11} by J_{10} so that one can use better estimates and deduce $\tau(n) = O(n^{11/2+\epsilon})$, for any $\epsilon > 0$.

Given the fact that there were others who were also trying to unravel some of the cryptic passages of Selberg's 1965 paper, I took the opportunity to indicate this proof in [Mu85]. The reader can also find a more friendly exposition in my recent monograph with V. Kumar Murty [MM13] amplifying the mathematical legacy of Srinivasa Ramanujan.

In early 1982, I fell ill with typhoid fever. Dr. Jotwani of TIFR gave me a regimen of antibiotics and I must say the one person that nursed me back to health through that difficult period was Srinivasan. He would bring me food from the cafeteria and force me to eat it. The antibiotics essentially destroyed any trace of appetite I had and I was also losing weight and strength. After about two weeks, I returned to normal health.

It was after this episode that I suggested to Srinivasan that we should do some joint work. We discussed many things but no formal paper materialized at that time, partly due to Srinivasan's very high standards as to what constitutes a paper. But on my later visits to TIFR, we continued our earlier discussions and managed to write two very interesting papers. One was about counting finite groups of squarefree order [MuSri87a] and the other was on exceptional sets to Artin's primitive root conjecture [MuSri87b]. I believe this latter paper caught the attention of Heath-Brown who subsequently improved upon my earlier work with Rajiv Gupta.

The paper on counting groups evolved over time, but it took me a while to interest Srinivasan in the problem. The question is simple. Up to isomorphism, how many groups of order n are there? A trivial estimate is n^{n^2} since that is the total number of all possible multiplication tables one can write down. Using the classification of simple groups, Neumann [Neu69] showed that this can be improved to

$$n^{c\log^3 n},$$

for some positive constant c. Greater precision emerges if one were to study subclasses of groups. For example, there is an extensive literature on the enumeration of p-groups, with p prime. These questions fascinated Kumar and myself when we were undergraduates and we began a series of papers that applied methods of analytic number theory to study such questions. For instance, if n is squarefree, the number of groups of order n is substantially smaller and in [MM84], we showed that this number is $\leq \varphi(n)$, where φ is Euler's function. In our paper with Erdös [ErMM87], we asked if this could be improved to $o(\varphi(n))$. This is the question Srinivasan and I answer in [MuSri87a]. In fact, we show the sharper result that the number is

$$\ll \frac{\varphi(n)}{(\log n)^{A\log\log n}},$$

for some constant $A > 0$.

In [MuSri87b], we studied the exceptional set to the Artin primitive root conjecture. Artin conjectured that for any $a > 1$, which is not a perfect square, there are infinitely many primes p for which $a \pmod p$ is a primitive root. We showed that the set of such $a \leq x$ for which Artin's conjecture is false is at most $O(\log^6 x)$. These results have been subsequently improved and refined.

In all my conversations with Srinivasan, I found him sharp and insightful. We had an after dinner seminar each day on the twin prime conjecture. The seminar was based on an old paper of Bombieri and each of us took turns to lecture on parts of this paper. It was at this time that I discovered Srinivasan had drawers full of notebooks in which he had studied many papers. It seems the way he did this was by copying the papers into his notebook, word for word. In those days, xeroxing was expensive and Srinivasan told me that this was the method used by others too like C. P. Ramanujam. Only recently did the benefits of this after-dinner study materialise in a tangible paper. My most recent work on twin primes and the parity problem (with Akshaa Vatwani) [MuVa17] has roots in my 1981 after-dinner conversations with Srinivasan.

Another related problem close to Srinivasan's heart was the Goldbach conjecture. As is well-known, the ternary Goldbach problem was solved by I.M. Vinogradov using the circle method. His approach to the binary Goldbach problem fails in its treatment of the "minor arcs". Srinivasan [Sri80] had a way of decomposing the minor arcs into two parts. He had a brilliant idea of using the large sieve inequality to show that one of these parts is negligible.

With Srinivasan in Bangalore in 2003

The last time I met Srinivasan was in Bangalore in 2003 at the conference honoring the 70th birthday of Professor Ramachandra. Srinivasan was a mathematician of the highest order and if he had written more, number theory would be that much richer today. What after all is the legacy we leave behind but for the written word. Our spoken words, our personalities all disappear like a lost fragrance in the vast ocean of time. It is only the written word that survives, which is but a faint shadow of what we have thought, of what we were and of what we have learned. Yet, it is the only means by which we can transfer the torch of knowledge to the next generation and after that to posterity.

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