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SOME RESULTS WHICH DESERVE TO BE WIDELY KNOWN

K.ALLADI, G.E. ANDREWS and A.BERKOVICH (MR 2063196 (2005g:11198)).
Let $(x)_\infty = \pi_{n=0}^{\infty}(1 - xq^n)$. A special case of the q-binomial theorem is

$$(-Aq)_\infty = \sum_i \; A^i \; \frac{q^{T_i}}{(q)_i}$$

where $T_i = \frac{1}{2}i(i+1)$ and $(x)_i = (x)_\infty/(xq^i)_\infty$.

A special case of the q-Chu-Vandermonde summation formula is

$$(-Aq)_\infty \; (-Bq)_\infty = \sum_{i,j,ij} \; A^{i+ij} \; B^{j+ij} \; \frac{q^{T_{i,j,ij}+T_{ij}}}{(q)_i \; (q)_j \; (q)_{ij}}.$$

Here and in the remaining equations ij is a single variable and not the product iXj. The triple product indentity is

$$(-Aq)_\infty(-Bq)_\infty(-Cq)_\infty = \sum_{i,j,k,ij,ik,jk} A^{i+ij+ik} \; B^{j+ij+jk} \; C^{k+ik+jk} \; X$$

$$X\frac{q^{T_s+T_{ij}+T_{ik}+T_{jk-1}(1-q^i+q^{i+jk})}}{(q)_i \; (q)_j \; (q)_k(q)_{ij}(q)_{ik} \; (q)_{jk}}.$$

where $s = i + j + k + ij + ik + jk$.

The authors discover a remarkable expansion of

$$(-Aq)_\infty \; (-Bq)_\infty \; (-Cq)_\infty \; (-Dq)_\infty.$$

Their expansion reads

$$\sum_{\substack{i,j,k,l,Q,\\ ij,ik,il,jk,jl,kl}} A^{i+ij+ik+il+Q} \; B^{j+ij+jk+jl+Q} \; C^{k+ik+jk+kl+Q} \; X$$

$$XD^{l+il+jl+kl+Q} \; X$$

$$X\frac{q^{T_\tau+T_{ij}+T_{ik}+T_{il}+T_{jk}+T_{kl}+T_{kl}+4T_{Q-1-jk-jl-kl+3Q+2Q\tau}}}{(q)_i\ (q)_j\ (q)_k\ (q)_l\ (q)_{ij}\ (q)_{ik}\ (q)_{il}\ (q)_{jk}\ (q)_{jl}\ (q)_{kl}\ (q)_Q}$$

$$X\left[(1-q^i)+Q^{i+jk+jl+Q}\ (1-q^j)+q^{i+jk+jl+Q+j+kl}\right]$$

where $\tau = i+j+k+l+ij+ik+il+jk+jl+kl$.