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HONORARY FELLOWS

The following eight mathematicians have been elected as HONORARY FELLOWS of the HARDY-RAMANUJAN SOCIETY.

(1) DANIEL ALAN GOLDSTON and

(2) CEM YALCM YILDIRIM.

The two have together proved that if

$$p_1 = 2, p_2 = 3, p_3 = 5, \dots$$

is the sequence of all primes then

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0$$

(3) YOICHI MOTOHASHI for his work

$$\int_0^T \zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + \Omega_\pm(T^{\frac{1}{2}})$$

where $P_4(x)$ is a certain polynomial of degree 4, with real coefficients.

(4) R.C.BAKER, (5) G.HARMAN and (6) JANOS PINTZ for their work

$$p_{n+1} - p_n < p_n^{\frac{1}{2}+\frac{1}{40}} \ (n \geq n_0)$$

where as before $p_1 = 2, p_3 = 3, p_4 = 5, \dots$ is the sequence of all primes. Also they prove

$$\pi(x+n) - \pi(x) \geq \frac{9}{100} \left(\frac{h}{\log x}\right)$$

for all $h \geq x^{\frac{1}{2}+\frac{1}{40}}$ and all $x \geq x_0$ (which is computable).

(7) WOLFGANG M. SCHMIDT for his work

$$|\alpha - \zeta| \geq \frac{C(\alpha,\epsilon,h)}{(H(\zeta))^{h+1+\epsilon}}$$

where α is an algebraic number of degree $n(n \geq 1)$ and $\zeta(\neq \alpha)$ is an algebraic number of degree $h(h \geq 1), \epsilon > 0$ is any arbitrary constant. $H(\zeta)$ denotes the maximum of the absolute values of the Polynomial (with content 1) with integer coefficients of which ζ is a root. Of course the constant $C(\alpha,\epsilon,h)(> 0)$ cannot be made explicit by the method. (Perhaps that needs an entirely different method.) Professor W.M.SCHMIDT has applied this to many diophantine questions. At present the only thing known about $C(\alpha,\epsilon,h)$ is that it exists and nothing more.

(8) N.I.ZAVOROTNYI for his result

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = \frac{T}{2\pi^2} P_4(\log T) + O_\epsilon(T^{\frac{2}{3}+\epsilon})$$

where $T \geq 100$ and $P_4(x)$ is a certain monic polynomial of degree 4 with real coefficients.

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RESULTS WHICH DESERVE TO BE WIDELY KNOWN

(1) CEM YAL CM YILDIRIM and DANIEL ALAN GOLDSTON have proved the following result,

$$\lim_{n\to\infty}\inf \frac{p_{n+1}-p_n}{\log p_n}=0$$

where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ is the sequence of all primes. This result was expected for many many decades, but this glorious achievements is available only now (see Proc. Japan Acad.Ser A Math. Sci. 82(2006) No.4, 61-65) see also Math.Rev. MR 22222 13 (2007 a:11135) 11 N05 11N 36 (Review by D.R.HEATH-BROWN. See also small gaps between primes. I 'Preprint, arxiv. org/abs/math/0504336).

(2) K.SOUNDARARAJAN has proved the following result. Let k be any positive constant and

$$M_k(T)=\int_0^T |\zeta(\frac{1}{2}+it)|^{2k}dt \; T\geq 100.$$

Assuming Riemann Hypothesis namely

$$\zeta(s)=\sum_{n=1}^{\infty}(n^s-\int_n^{n+1}\frac{du}{u^s})+\frac{1}{s-1}\neq 0(s=\sigma+it$$

(for all $\sigma > \frac{1}{2}$), we have

$$T(\log T)^{k^2} \ll_k \; M_k(T) \; \ll_{k,\epsilon} \; T(\log T)^{k^2+\epsilon}$$

for all $\epsilon > 0$. (The constant involved in $\ll_k$ depends only on k and in $\ll_{k,\epsilon}$ depends only on k and ϵ). For some important results by other authors see the references at the end of his paper K.SOUNDARAJAN, Moments of the Riemann zeta-function (to appear).

(3) the following result due to R.BALASUBRAMANIAN is well-known (ref. An improvement of a theorem of Titchmarsh on the mean-square of $|\zeta(\frac{1}{2}+it)|$, Proc.London Math. Soc., (3) 36(1978), 540-576)

$$\frac{1}{2\pi}\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt=\frac{T}{2\pi}\log\frac{T}{2\pi}+(2\gamma-1)\frac{T}{2\pi}+O(T^{\frac{1}{3}}) \; (T\geq 100)$$

(he proves also the result with $\frac{1}{3}$ replaced by smaller constants). Put

$$E(T)=\frac{1}{2\pi}\int_0^T |\zeta(\frac{1}{2}+it)|^2 \, dt-\left\{\frac{T}{2\pi}\log\frac{T}{2\pi}+(2\gamma-1)\frac{T}{2\pi}\right\}.$$

BALASUBRAMANIAN's result is $E(T) = O(T^{\frac{1}{3}})$. YUK-KAM LAU and KAI-MAN TSANG prove that

$$E(T) = \Omega((T \log T)^{\frac{1}{4}} (\log\log T)^{\lambda} (\log\log\log T)^{-\mu})$$

where $\lambda = \frac{3}{4}(2^{\frac{4}{3}} - 1)$ and $\mu = \frac{5}{8}$. (to appear).

(4) ALEKSANDAR IVIC and PATRICK SARGOS have proved

$$\int_1^T E^3(t)dt = B_1 T^{\frac{7}{4}} + O_\epsilon(T^{\beta_1+\epsilon})$$

and also

$$\int_1^T E^4(t)dt = C_1 T^2 + O_\epsilon(T^{\gamma_1+\epsilon})$$

where $\epsilon > 0$ is arbitrary, B_1, C_1 are positive constants, $\beta_1 = \frac{5}{3}$ and $\gamma_1 = \frac{23}{12}$. Also they prove some important results on the error terms in the circle problem and divisor problems. For example the following result is worth noting. Let $P(x)$ denote the number of lattice points inside or on a circle of radius $\sqrt{x}$ minus πx. Then

$$\int_1^X (P(x))^4 dx = C_2 X^2 + O(X^{\gamma_1+\epsilon})$$

where $C_2 > 0$ is a constant. (Ref: on the higher moments of the error-term in the divisor problem, Illinois Journal of Mathematics, Vol. 51, No 2, Summer 2007 353-377).

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S.S. PILLAI REMEMBERED (05-04-1901 TO 31-08-1950)

K.RAMACHANDRA wishes to recall here an incident related to his friend L.CHANDRASEKHA (24-09-1934 TO 17-01-2006). He was an Amateur physicist. He had borrowed volume 12(1940) of the Proc. Indian Acad. Sci. from RAMAN RESEARCH INSTITUTE. He had been to visit him in his house around 1945 and he noticed the result g(6)=73, Proc. Indian Acad. Sci. Vol. 12 (1940) pp.30-40). He was curious to know what this means and came to know that S.S.PILLAI (using the great work of I.M.VINOGRADOV) had proved "the following result". let $k \geq 2$ and denote by $A(k)$ the quality $2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$. The equation

$$n = \sum_{j=1}^{A(k)} m_j^k \ (n \geq 1 \text{ any integer,} \mathrm{m_j} \text{integers } \geq 0) \tag{1}$$

is solvable for all n and there is an integer $n_0 \geq 1$ such that

$$n_0 = \sum_{j=1}^{A(k)-1} m_j^k$$

is not solvable. This is known as WARING'S PROBLEM. PILLAI had proved this for $6 \leq k \leq 100$. For general k he had to assume some condition like

$$| \left(\frac{3}{2}\right)^k - \text{nearest integer } |^{k^{-1}} \geq \frac{7}{8}. \tag{2}$$

We know that this is true (Thanks to the work of A.THUE, C.L.SIEGEL, F.J. DYSON, Th.SCHNEIDER, K.F.ROTH, D.RIDOUT and K.MAHLER) for all $k \geq k_0$ for some integer k_0. We know that it exists. But unfortunately we do not know any such value of k_0. We do not even know 'HOW MANY INTEGERS k DO NOT SATISFY (2)! " although we know that their number is a certain fixed positive integer. We are content (to start with) to get some upper bound for this integer. But even this is not known. Since the number of exceptions (in k) to WARING'S PROBLEM depends on such questions, the present status of this problem is

WARING'S CONJECTURE IS CORRECT FOR ALL $k \geq k_1$, WHERE k_1 IS SOME FIXED POSITIVE INTEGER (BUT WE DO NOT KNOW ANY SUCH NUMBER k_1. WE KNOW ONLY THAT k_1 EXISTS)

For some small values of k the problem posed some difficult problems. Thus $k = 6$: $g(6) = 73$ due to S.S.PILLAI, $k = 5$: ($g(5) = 37$) DUE TO C.J.RUN, $k = 4$: The two results $g(4) \leq 21$, $g(4) \leq 20$ were published in HRJ in two of its volumes (namely 2 & 8) by R.BALASUBRAMANIAN and the final result $g(4) = 19$ was solved finally jointly by R.BALASUBRAMANIAN, J.-M.DESHOUILLERS and F.DRESS. $k = 2$ and $k = 3$ are well known. PILLAI'S method is known as the method of STEEPEST ASCENT. For those values of k for which the conjecture is true it is customary to denote this value of $A(k)$ by $g(k)$. If for a certain value or values of k the conjecture is not true (possibly), alternative values of $g(k)$ was given by PILLAI.