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# BOOK REVIEW

**A PASSAGE TO INFINITY: Medieval Indian Mathematics from Kerala and its impact**, by George Gheverghese Joseph, Sage Publications India Private Limited, 2009, 220p. With bibliography and index. ISBN 978-81-321-0168-0.

**Reviewed by M. Ram Murty, Queen's University.**

It is well-known that the profound concept of zero as a mathematical notion originates in India. However, it is not so well-known that infinity as a mathematical concept also has its birth in India and we may largely credit the Kerala school of mathematics for its discovery. The book under review chronicles the evolution of this epoch making idea of the Kerala school in the 14th century and afterwards.

Here is a short summary of the contents. After a brief introduction, chapters 2 and 3 deal with the social and mathematical origins of the Kerala school. The main mathematical contributions are discussed in the subsequent chapters with chapter 6 being devoted to Madhava's work and chapter 7 dealing with the power series for the sine and cosine function as developed by the Kerala school. The final chapters speculate on how some of these ideas may have travelled to Europe (via Jesuit missionaries) well before the work of Newton and Leibniz. It is argued that just as the number system travelled from India to Arabia and then to Europe, similarly many of these concepts may have travelled as methods for computational expediency rather than the abstract concepts on which these algorithms were founded.

Large numbers make their first appearance in the ancient writings like the Rig Veda and the Upanishads. They also appear in the Ramayana and the Mahabharata. The problem of infinity as a mathematical idea initially appears in Brahmagupta's Brahmasphutasiddhanta in which he raises the question of what is the value of $1/0$. This question is answered in the 12th century by Bhaskaracharya who correctly deduces that it is infinity by an ingenious limit process. Indeed, by that time, the rules for operating with fractions were clear and Bhaskara proceeds to assert that since $\frac{1}{(1/2)} = 2$, and $\frac{1}{(1/3)} = 3$ and so on, it is clear that $\frac{1}{(1/n)}$ is $n$ and as $n$ tends to infinity, he deduces that $1/0$ is infinity. This profound discovery stands on par with the discovery of zero and in fact links the two discoveries. It opens the way for the theory of limits and infinite series which was taken up in greater detail by Madhava and the Kerala school.

This book delineates the contributions of the 14th century mathematician Madhava and his successors to the theory of limits and infinite series which are the rudimentary notions for the development of calculus. Thanks to new research in the 20th century, we know that Madhava had derived what are now called Taylor series of the classical trigonometric functions like sine and cosine several centuries before the European mathematicians. For instance, using ingenious geometric methods, he found that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ,$$

a series often attributed to Leibniz in the 18th century. Indeed, Madhava begins by noting that the problem of determining the value of $\pi/4$ is the same as that of determining the arc length of the segment of the circle with unit radius subtended by an angle of 45°. Madhava subdivides the segment $AB$ into $n$ equal parts $A_0A_1, \ldots, A_{n-1}A_n$

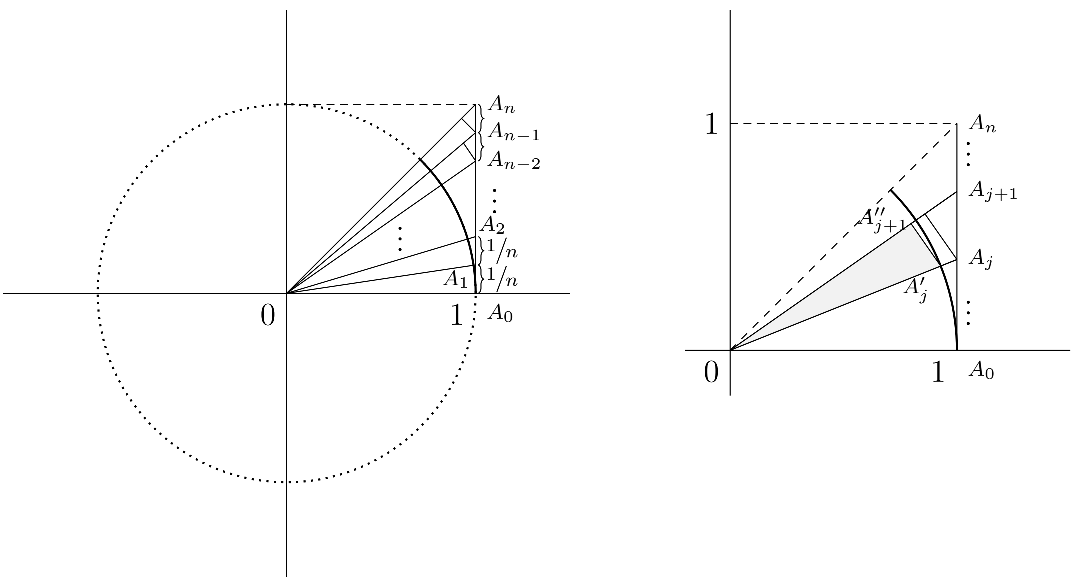


FIGURE 1. Madhava's Series for $\pi/4$

where $A_0 = A$ and $A_n = B$. For $0 \leqslant j \leqslant n-1$, we set $A'_j$ to be the point of intersection of the circle and the line $OA_j$ and $A''_{j+1}$ to be the foot of the perpendicular from $A_j$ to the line $OA_{j+1}$ (see Figure 1). Now, by approximating the arc by lengths of the sides $A'_jA''_{j+1}$ of the right angled triangles $OA''_{j+1}A'_j$, Madhava finds that the arc length is approximated by

$$\text{(1)} \qquad \frac{1/n}{1+(1/n)^2} + \frac{1/n}{1+(2/n)^2} + \frac{1/n}{1+(3/n)^2} + \cdots + \frac{1/n}{1+((n-1)/n)^2} + \frac{1/n}{1+(n/n)^2}.$$

The modern student of calculus will immediately recognize that this is the Riemann sum of the integral $\int_0^1 \frac{\mathrm{d}x}{1+x^2}$. But not having the calculus, Madhava uses geometric series to deduce the final result. He was aware that, for $0 < r < 1$,

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots .$$

Now, he expands each of the terms in (1) as an infinite series:

$$\frac{1}{1+(1/n)^2} = 1 - \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^4 - \left(\frac{1}{n}\right)^6 + \cdots$$
$$\frac{1}{1+(2/n)^2} = 1 - \left(\frac{2}{n}\right)^2 + \left(\frac{2}{n}\right)^4 - \left(\frac{2}{n}\right)^6 + \cdots$$
$$\frac{1}{1+(3/n)^2} = 1 - \left(\frac{3}{n}\right)^2 + \left(\frac{3}{n}\right)^4 - \left(\frac{3}{n}\right)^6 + \cdots$$

There are $n$ such rows and each being multiplied by $1/n$, Madhava adds up the columns using the formula

$$\frac{1}{n^{k+1}}\left[1^k + 2^k + 3^k + \cdots + (n-1)^k\right] \approx \frac{1}{k+1}.$$

Such formulas were already there in the work of Bhaskaracharya and so this was a natural application of the idea. It is now clear how the desired formula for $\pi/4$ emerges from this analysis. This remarkable chapter in the history of Indian mathematics is the focus of attention in Chapter 6 of Joseph's book. There is however a small gap in this derivation and that relates to the error term in the above formula for the sum of the $k$th powers of the natural numbers. Since the explicit formulas for small values of $k$ were written down in many of the mathematical texts of Medieval India, we may assume that Madhava was aware that the sum is $\frac{1}{k+1} + \frac{1}{2n} + O(\frac{1}{n^2})$ and the analysis now goes through with complete rigor. Madhava derives in a similar way, the now familiar infinite series for the sine and the cosine function.

What emerges from Joseph's book is a new understanding of an unbroken continuity of the Indian mathematical tradition beginning with Aryabhata to the modern period. Earlier, due to scanty historical research, we had the impression that the Indian discoveries were sporadic and isolated. But the findings of the work of Madhava and his school changes all that. It seems that these findings first came to light in 1834 when Charles Whish published a paper in the *Transactions of the Royal Asiatic Society* entitled "On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four sastras, the Tantrasangraham, Yukti Bhasha, Caruna Paddhati, and Sadratnamala". However, these findings did not seem to make it to the history books, largely because many did not read the Royal Asiatic Society Journal and partly because there was a European bias that fundamental notions of calculus could not have been discovered by an Indian. Indeed, the noted historian of mathematics, David Pingree wrote: "One example I can give you relates to the Indian Mādhava's demonstration, in about 1400 A.D., of the infinite power series of trigonometrical functions using geometrical and algebraic arguments. When this was first described in English by Charles Matthew Whish, in the 1830s, it was heralded as the Indians' discovery of the calculus. This claim and Mādhava's achievements were ignored by Western historians, presumably at first because they could not admit that an Indian discovered the calculus, but later because no one read anymore the Transactions of the Royal Asiatic Society, in which Whish's article was published. The matter resurfaced in the 1950s, and now we have the Sanskrit texts properly edited, and we understand the clever way that Mādhava derived the series without the calculus; but many historians still find it impossible to conceive of the problem and its solution in terms of anything other than the calculus and proclaim that the calculus is what Mādhava found. In this case the elegance and brilliance of Mādhava's mathematics are being distorted as they are buried under the current mathematical solution to a problem to which he discovered an alternate and powerful solution." (See p. 562 of D. Pingree, *Hellenophilia versus the History of Science*, Isis 83(4) (1992), 554–563.) Moreover, Whish's paper appears at the height of colonial rule and consistent with the phenomenon of "orientalism" (as noted by the historian Edward Said), any contribution from a "subject nation" was deliberately ignored or undervalued. This applied equally to contributions from Africa or other Asiatic nations.

Thus, given this glaring omission in the historical account of the evolution of modern mathematics by European scholars, Joseph's book is a valuable document. Students within India and outside of India can trace the development of these fundamental ideas and their impact on modern civilization. But this book merely scratches the surface of what is clearly a new field of research. As Joseph writes, "This is only a short account of a vast tradition and as such only a few landmarks on the highway have been touched. Explorative studies have been carried out only on a small percentage of the mass of manuscripts that have come down to us from the past. An enormous amount of primary material lies unexplored in various repositories." Apparently, there are as many as 3473 science texts in Sanskrit and 12,244 science manuscripts from more than 400 repositories in Kerala and Tamil Nadu and so the task is immense for the research historian of mathematics. This book is therefore the first step towards the completion of this task.